1. The governing equations of electromagnetic phenomena

For the understanding of natural phenomena in most cases it is unnecessary to take into consideration the atomic and molecular structure of the material, because the time interval and distance of the phenomena or process is much larger than the characteristic time interval and distance of atomic phenomena. In this case the phenomenological description can be used. Phenomenological theories provide a simplified approximate description of the material world, their advantage is that they provide a simple and exact mathematical description of the most important processes taking place on a macroscopic scale. An inevitable result of the phenomenological description is that we have to introduce the material properties (elastic constants, viscosity, thermal conductivity, dielectric constant, magnetic susceptibility, etc.). These material properties substitute those properties (atomic structure, crystal structure), which cannot be examined on macroscopic scale. The theory of electromagnetic phenomena developed based on the above is called phenomenological electrodynamics and since it is sufficient for geophysical applications we discuss only phenomenological electrodynamics in this book.

There are four basic equations, called Maxwell equations, which govern electromagnetic phenomena. The so called local forms of these equations are:

$$\text{rot}\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \tag{1.1}$$

$$\text{rot}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.2}$$

$$\text{div}\,\vec{D} = \rho \tag{1.3}$$

$$\text{div}\,\vec{B} = 0 \tag{1.4}$$

Here rot (curl) is the so called vortex density, $\vec{H}$ is vector of the magnetic field strength, $\frac{\partial \vec{D}}{\partial t}$ is the time derivative of the electric displacement vector $\vec{D}$, $\vec{E}$ is the electric field strength, $\frac{\partial \vec{B}}{\partial t}$ is the time derivative of the magnetic induction vector $\vec{B}$, div is the so called source density, ρ is the charge density and $\vec{j}$ is the current density vector, which consist of the convection current density and the conduction current density, however in geophysical applications convection current has no significance, therefore in the followings $\vec{j}$ will denote the conduction current density.

1.1 Boundary conditions

The Maxwell equations are coupled linear differential equations applying locally at each point in space-time (x, t) and for solving them boundary conditions are needed. Using Stokes theorem based on Eq. 1.1 it can be seen that the tangential component of $\vec{H}$ is continuous across the interface:

$$\vec{H}_t^{(1)} = \vec{H}_t^{(2)}. \tag{1.5}$$

Similarly, from (1.2) we can derive that:

$$\vec{E}_t^{(1)} = \vec{E}_t^{(2)}. \tag{1.6}$$

With the help of the Gauss-Ostrogradsky theorem, from Eq. 1.3 we can derive for $\vec{D}$ vector's normal component the following boundary condition across the interface:

$$D_n^{(2)} - D_n^{(1)} = \sigma,$$

where σ is the surface charge density. Based on Eq. 1.4 we can set the boundary condition that the component of the magnetic induction vector is continuous across the interface:

$$B_n^{(1)} = B_n^{(2)}.$$

1.2 Material equations

$\vec{D}$ *and* $\vec{E}$ in the Maxwell equations and the $\vec{B}$ *and* $\vec{H}$ are not independent. In vacuum we can define the

$$\vec{D} = \varepsilon_0\vec{E} \;\text{ and }\; \vec{\mathrm{B}} = \mu_0\vec{H} \qquad (1.7)$$

equations, where $\varepsilon_0 = 8{,}85 * 10^{-12}\,\frac{As}{Vm}$ is the dielectric constant or permittivity of vacuum and $\mu_0 = 4\pi * 10^{7}\,\frac{Vs}{Vm}$ is the magnetic permeability of vacuum.

If $\vec{P}$ is the electric- $\vec{\mathrm{M}}$ is the magnetic polarization vector (which mean the electric and magnetic dipole moment per unit volume respectively), then we can use the following definitions for $\vec{D}$ and $\vec{B}$:

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \text{ and } \vec{\mathrm{B}} = \mu_0\vec{H} + \vec{M}. \qquad (1.8)$$

The relationship of $\vec{P}$ and $\vec{E}$ vectors and the relationship of $\vec{M}$ and $\vec{H}$ vectors are investigated by taking into consideration the atomic structure of the material by the electron theory, statistical mechanics and quantum physics. In phenomenological electrodynamics we simplify this relationship and we do not deal with the atomic nature of things. It is the easiest to assume a linear relationship:

$$\vec{P} = \chi\varepsilon_0\vec{E} \text{ and } \vec{M} = \chi\mu_0\vec{H}\;, \qquad (1.9)$$

where χ_e and χ_m are the electric and magnetic susceptibility of the medium respectively, or based on Eq. (1.8):

$$\vec{D} = \varepsilon\vec{E} \qquad (1.10)$$

and

$$\vec{\mathrm{B}} = \mu\vec{H}\;, \qquad (1.11)$$

where $\varepsilon = \varepsilon_0(1+\chi)$ and $\mu = \mu_0(1+\chi)$ are the permittivity and the permeability of the medium respectively. The dielectric properties of the medium characterized by susceptibilities and permeabilities are taken as known material constants in phenomenological electrodynamics or in case of inhomogeneous medium, as a function of the space coordinates. However, these "material constants" are temperature dependent and often even depend on the frequency of the electromagnetic field. (This phenomenon can be explained by taking into consideration the inner structure of the material). The linearity of Eq. 1.10 and 1.11 are usually adequately fulfilled, which means that ε and μ are independent from the field strengths. For ferroelectric and ferromagnetic materials, the relationship of $\vec{D}-\vec{E}$ and $\vec{B}-\vec{H}$ cannot be given by a linear function, and it is not even a single valued function (hysteresis).

For materials which show anisotropy on macroscopic scale, Eq. (1.10-1.11) are not valid, therefore the following more general material equations are used instead:

$\vec{D} = \underline{\underline{\varepsilon}}\,\vec{E}$ and $\vec{B} = \underline{\underline{\mu}}\,\vec{H}$,

where $\underline{\underline{\varepsilon}}$ is the dielectric, $\underline{\underline{\mu}}$ is the magnetic permeability tensor. In this case usually the $\vec{D}$ *and* $\vec{E}$, $\vec{B}$ *and* $\vec{H}$ vectors are not parallel.

In a conductive medium the current density is usually not pre defined, but is defined by the strength of the electric field. For homogenous and isotropic bodies, this relationship is given by the differential Ohm's Law:

$$\vec{J} = \gamma \vec{E} \ , \qquad (1.12)$$

where γ is the electrical conductivity. This material property also depends on temperature, frequency and on other parameters connected to the inner structure of the material. However, phenomenological electrodynamics does not deal with these effects, γ electrical conductivity is considered a known constant.

1.3 The completeness of the Maxwell equations

The system of equations (1.1) – (1.4) describe the electromagnetic field. Therefore, it is very important whether these set of equations have a solution and whether the solution is clearly defined or not. In this sense, the first question is whether the number of independent equations equal the number of unknowns in this system of equations. It is obvious that the set of equations (1.1) – (1.4) are underdetermined.

Taking into consideration the (1.10) – (1.12) material equations and assuming the ε, μ, γ material properties to be known, we now have 15 unknown functions ($\vec{D}, \vec{E}, \vec{B}, \vec{H}, \vec{J},$) to be determined by 17 equations. Therefore, the set of equations is overdetermined. However, it can be proved that equations (1.3) and (1.4) are not independent from (1.1) and (1.2).

The conservation of electric charge is a natural law, which is valid independently form the laws of the electromagnetic field. In phenomenological electrodynamics the conservation of electric charge is described by the continuity equation of electric charge:

$$\frac{\partial \rho}{\partial t} + div\,\vec{J} = 0 \ . \qquad (1.13)$$

Let's take the divergence of Eq. (1.1)! Then because of div rot $\vec{H} = 0$

$$\frac{\partial}{\partial t} + div\,\vec{D} + div\,\vec{J} = 0 \ , \qquad (1.14)$$

where the order of derivation by the space coordinates and time have been interchanged. Forming the difference between (1.13) and (1.14) we get

$$\frac{\partial}{\partial t}(div\,\vec{D}-\rho)=0 \qquad (1.15)$$

or in a different way:

$$div\vec{D}-\rho=C\ ,$$

where C is independent from time. If at t=0 we set the initial conditions in such a way that C=0, then according to (1.15) at any given time:

$$div\vec{D}-\rho=0\ .$$

However, this means that the Maxwell equation (1.3) does not state an independent law from (1.1), just a rule for setting the initial conditions.

In a similar way, if we take the divergence of (1.2) we get

$$\frac{\partial}{\partial t}\left(div\,\vec{B}\right)=0\ .$$

So if at t=0 we write the initial conditions as:

$$div\,\vec{B}=0\ ,$$

then the equation is fulfilled at any given time. Which means that the Maxwell equation (1.4) is not independent from (1.2). From these we can see that the (1.1) - (1.4) Maxwell equations with the (1.10) – (1.12) material equations form a complete system of equations.

The exact solution of the Maxwell equations is set by the initial and boundary conditions. When setting these conditions, we have to take into consideration the (1.3), (1.4) equations and the boundary conditions presented earlier.

1.4 The special phenomena of electrodynamics

With the Maxwell equations all the electromagnetic phenomena can be described in a theoretical way. There are however some phenomena, which do not require the usage of the (1.1) – (1.4) equations in their general form. Because of the high number of these phenomena, it is common to divide electrodynamics into different chapters.

We deal with statics if the physical quantities are constant with time, the charges are in permanent magnetic state and there is no flowing current. In the Maxwell equations then $\vec{J}=0$ and $\frac{\partial}{\partial t}=0$.

The basic equation of <u>electrostatics:</u>

$$\text{rot}\,\vec{E}=0 \qquad \vec{H}=0,\ \vec{B}=0$$

$$\text{div}\,\vec{D}=0 \qquad \vec{D}=\varepsilon\vec{E}$$

The basic equations of <u>magnetostatics:</u>

$$\text{rot}\,\vec{H}=0 \qquad \vec{E}=0,\ \vec{D}=0$$

$$\text{div}\,\vec{B}=0 \qquad \vec{B}=\mu\vec{H}$$

We talk about stationary currents when the flowing currents in the conducting medium are independent from time. It is easy to see that in this case there is no volume charge density in the Maxwell equations. The continuity equation of charge:

$$\frac{\partial \rho}{\partial t} + div\,\vec{J} = 0\ ,$$

and using the differential Ohm's law $\vec{J} = \gamma\vec{E}$, based on Eq(1.3) we get

$$\frac{\partial \rho}{\partial t} + \frac{\gamma}{\varepsilon}\rho = 0\ .$$

If we assume that inside the conducting media at t=t_0 there is ρ_0 volume charge density, then by solving the above equation we get

$$\rho(t) = \rho_0 e^{-\frac{t}{\tau}}\ ,$$

where $\tau = \frac{\gamma}{\varepsilon}$ is the relaxation time of the medium. This equation indicates that the volume charge density decreases exponentially. (During the relaxation time it decreases by $\frac{1}{e}$). This can be explained easily: the charges inside the conductor can move and because of their repelling effect they get to the surface of the conductor. Relaxation time describes this process. Rocks that are important from a geophysical point of view, limestone has the lowest conductivity ($\gamma \approx 10^{-3}\frac{1}{\Omega m}$) which results in $\tau \approx 10^{-10}s$. For other rocks τ is even smaller.

So even if there are volume charges in the conductor, they get to the conductor's surface in a very short time and their effect only last for the relaxation time. After a longer period, the volume charge density is zero. Therefore, in case of stationary processes the basic equations can be written as:

$$\text{rot}\ \vec{H} = \vec{J}, \qquad rot\ \vec{E} = 0,$$

$$\text{div}\ \vec{D} = 0, \qquad div\ \vec{B} = 0$$

$$\vec{D} = \varepsilon\vec{E}, \qquad \vec{B} = \mu\vec{H}, \qquad \vec{J} = \gamma\vec{E}$$

We talk about Quasi-Stationary processes when the displacement current density is negligible beside the current density flowing in the medium. The condition for this can be easily derived for phenomena that are time dependent by $e^{i\omega t}$. Let's assume that in the medium the electric field strength is $\vec{E} = \vec{E_0}\ e^{i\omega t}$. Then according to the differential Ohm's law:

$$\vec{J} = \gamma\vec{E_0}e^{i\omega t}\ ,$$

and the displacement current density

$$\frac{\partial \vec{D}}{\partial t} = i\omega t\ \vec{E_0}\ e^{i\omega t}\ .$$

The $\vec{J} \gg \frac{\partial \vec{D}}{\partial t}$ magnitude comparison lead to the $\omega \ll \frac{\gamma}{\varepsilon}$ relation.

This can also be written as $T \gg \tau$, where $T = \frac{2\pi}{\omega}$ is the period, τ is the relaxation time as described earlier. It is obvious that volume charges cannot be in the medium in this case either. The $\omega_0 = \frac{1}{\tau}$

cutoff frequency in case of rocks is the smallest for limestone $\omega_0 \approx 10^8 \frac{1}{s}$. However, this is a very high frequency, the condition $\omega \ll \omega_0$ is practically always fulfilled in geophysical application. The basic equations of the quasi-stationary currents' field:

$$\text{rot}\,\vec{H} = \vec{J}, \qquad rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\,\vec{D} = 0, \qquad div\,\vec{B} = 0$$

$$\vec{D} = \varepsilon \vec{E}, \qquad \vec{B} = \mu \vec{H}, \qquad \vec{J} = \gamma \vec{E}$$

For fields that are changing rapidly with time ($\omega \gg \omega_0$) the displacement current density in Eq.(1.1) needs to be taken into consideration as well. In this case for describing electromagnetic phenomena we use the system of Maxwell equations (1.1) – (1.4).

2. Electromagnetic potentials

The Maxwell equations are coupled partial differential equations. Their general solution cannot be written directly. Therefore, any method that simplifies the solution of the field equations is very useful. One of this method was the introduction of electromagnetic potentials.

Equation (1.4) can be trivially satisfied if we take the magnetic induction vector in the following form:

$$\vec{B} = rot\vec{A} \; , \tag{2.1}$$

where $\vec{A}(r,t)$ vector field for the time being is the unknown vector potential. With this (1.2) can be written as

$$rot\;(\vec{E} + \frac{\partial \vec{A}}{\partial t} = 0 \; .$$

This equation can be trivially satisfied, if

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -grad\;\Phi \; ,$$

where $\Phi(r,t)$ is the presently unknown scalar potential which is an arbitrary, continuous function that is differentiable. With the vector- and scalar potential the electric field can be given as:

$$\vec{E} = -grad\;\Phi - \frac{\partial \vec{A}}{\partial t} \quad . \tag{2.2}$$

It can be seen that with the four introduced scalar function $(\Phi, \vec{A})$, (2.1), (2.2) $\vec{E}$ and $\vec{B}$ i.e. six scalar fields can be defined.

For the unknown potentials we can derive relationships based on (1.1) and (1.3) and on the material equations (1.10) and (1.11). Assuming a homogenous medium ($\varepsilon\ and\ \mu\ are\ independent\ from\ location$) based on (1.1) we get the equation

$$\Delta\vec{A} - \varepsilon\mu\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J} + grad\;(div\vec{A} + \varepsilon\mu\frac{\partial \Phi}{\partial t}) \; , \tag{2.3}$$

where we used the

$$rot\;rot\;\vec{A} = grad\;div\;\vec{A} - \Delta\vec{A}$$

identity. Similarly based on (1.3) we get the equation

$$\Delta\Phi - \varepsilon\mu\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon} - \frac{\partial}{\partial t}(div\;\vec{A} + \varepsilon\mu\frac{\partial \Phi}{\partial t}) \; . \tag{2.4}$$

(2.3) and (2.4) are second order coupled linear partial differential equations of electromagnetic potentials. The solution of these mathematically is just as complex as the solution of the Maxwell equations. A significant simplification is possible if we examine the clear determination of potentials.

2.1 Gauge transformation, Lorentz gauge condition

It can be seen easily that the electromagnetic potentials with the (2.1), (2.2) equations are not clearly determined. Let's form the

$$\overrightarrow{A'} = \vec{A} + grad\,\chi \qquad (2.5)$$

new potential, where χ is an arbitrary function. Based on (2.1) we can see that the vector space calculated with this is as follows:

$$\overrightarrow{B'} = rot\,\overrightarrow{A'} = rot\,\vec{A} + rot\,grad\,\chi$$

and it equals $\vec{B} = rot\,\vec{A}$, since $rot\,grad\,\chi = 0$. So it clearly determines the vector potential (2.1) only to the extent of the gradient of an arbitrary χ function. However, the vector potential modified or transformed according to (2.5) produces the same B vector space.

Let's modify the scalar potential according to:

$$\Phi' = \Phi - \frac{\partial\chi}{\partial t} \ , \qquad (2.6)$$

then we get the

$$\overrightarrow{E'} = -grad\Phi' - \frac{\vec{A}'}{\partial t} = -grad\Phi - \frac{\partial\vec{A}}{\partial t}$$

field strength, so according to (2.2) $\overrightarrow{E'} = \vec{E}$.

The modification of the electromagnetic potentials according to (2.5), (2.6) is called gauge transformation. This transformation leaves the $\vec{B}$ and $\vec{E}$ fields unchanged, so the system of Maxwell equations are not affected by this transformation. In other words, the Maxwell equations are invariant under the gauge transformation.

The determination of potentials by (2.5), (2.6) is unclear, therefore additional restrictions need to be defined. These restrictions are given automatically by (2.3) and (2.4), because if we specify the equation

$$div\,\vec{A} + \varepsilon\mu\frac{\partial\Phi}{\partial t} = 0\ , \qquad (2.7)$$

then Eq. (2.3), (2.4) are simplified. Eq. (2.7) imposed on the potentials is called the Lorentz gauge condition.

2.2 Potential equations, retard potential

If the (2.7) Lorentz gauge condition is fulfilled, then Eq. (2.3) and (2.4) take the following forms:

$$\Delta\vec{A} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}\ . \qquad (2.8)$$

$$\Delta\Phi - \varepsilon\mu\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon}\ . \qquad (2.9)$$

These equations are inhomogeneous wave equations, or the so called d'Alembert differential equations. Having the Lorentz gauge condition, the potential equations become uncoupled thus the components of the vector potential and the scalar potential now can be determined independently from each other.

The solution of Eq. (2.9) is given by the sum of the general solution of the homogeneous equation and one particular solution of the non-homogenous equation. The solution of the homogenous equation (wave equation) will be discussed later. The particular solution can be written as follows.:

$$\Phi(x_1, x_2, x_3, t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(\vec{r}; t - \frac{R}{v})}{R} dx'_1 dx'_2 dx'_3 \,, \tag{2.10}$$

where $R = \left|\vec{r'} - \vec{r}\right|$, $v = \frac{1}{\sqrt{\varepsilon\mu}}$ and the integration needs to be extended to that V' part of the field where the charges are located.

According to the (2.10) expression the value of potential at point P of the field denoted by the vector $\vec{r}$ at t time can be determined by the summation (integration) of the unit potentials deriving from the $dQ = \rho dV'$ charges located in the dV' unit volumes that are situated around point P'.

$$d\Phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0}\frac{\rho(\vec{r}; t - \frac{R}{v})}{R} dV'$$

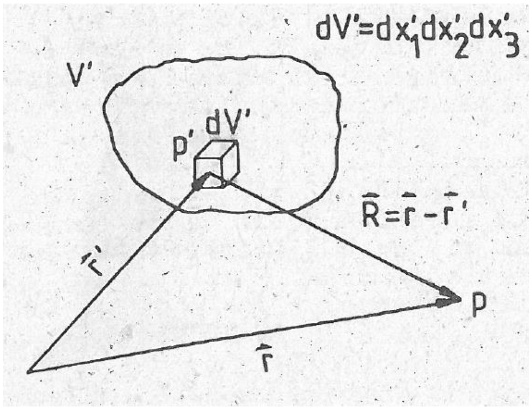

Figure 1.

The integration needs to be extended to that V' part of the field where the charges are located. At time t, the unit potential in point P is determined by the value of charge density of point P' not at t but at time $t' = t - \frac{R}{v}$. Since $\vec{R}$ is the distance between point P and P', v is the velocity of propagation of the electromagnetic effect, the $\frac{R}{v}$ difference between the two times equals the time, in which the electromagnetic effect gets to point P from P'.

Because of the electromagnetic effect's finite velocity of propagation, the effect of the charge density change in point P' occurs later ($t - t' = \frac{R}{v}$) in point P (this delay is called retardation). The scalar potential (2.10) takes into consideration this retardation, and that is why the solution of Eq. (2.9) is called the retarded potential.

The solution of Eq. (2.8) can be written similarly

$$\vec{A}(r, t) = \frac{\mu}{4\pi}\frac{\vec{J}(\vec{r}; t - \frac{R}{v})}{R} dV' \,. \tag{2.11}$$

Based on (2.10) and (2.11) the sources $\rho(\vec{r}, t')$, $\vec{J}(\vec{r}, t)$ are known, the retarded potentials and through (2.1), (2.2) the electromagnetic fields can be determined.

2.3 Electromagnetic potentials in conductors

The current- and charge density on the right side of the potential equations (2.8), (2.9) can be considered as the sources of the field, knowing these the solution can be written. However, for several electromagnetic problems the current density is unknown, it develops through (1.12) the differential Ohm's law closely connected to the electromagnetic field. Based on (1.12) and (2.2) the equation (2.3) can be written as:

$$\Delta\vec{A} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} - \gamma\mu\frac{\partial\vec{A}}{\partial t} = grad\,(div\vec{A} + \varepsilon\mu\frac{\partial\Phi}{\partial t} + \gamma\mu\Phi)\ . \tag{2.12}$$

In geophysical applications ρ volume charge density is not considered as a source, therefore $\rho = 0$ can be used. Adding $-\gamma\mu\frac{\partial\Phi}{\partial t}$ to both sides of Eq. (2.4) we get

$$\Delta\Phi - \varepsilon\mu\frac{\partial^2\Phi}{\partial t^2} - \gamma\mu\frac{\partial\Phi}{\partial t} = -\frac{\partial}{\partial t}(div\,\vec{A} + \varepsilon\mu\frac{\partial\Phi}{\partial t} + \gamma\mu\Phi)\ . \tag{2.13}$$

Based on these equations we can see that due to the Maxwell equations gauge invariance, for the electromagnetic potentials now it is advisable to set the condition as follows:

$$div\vec{A} + \varepsilon\mu\frac{\partial\Phi}{\partial t} + \gamma\mu\Phi = 0\ . \tag{2.14}$$

Then the equations (2.12) and (2.13) become uncoupled and the potential equations can be written as:

$$\Delta\vec{A} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} - \gamma\mu\frac{\partial\vec{A}}{\partial t} = 0\ . \tag{2.15}$$

$$\Delta\Phi - \varepsilon\mu\frac{\partial^2\Phi}{\partial t^2} - \gamma\mu\frac{\partial\Phi}{\partial t} = 0\ . \tag{2.16}$$

So in conducting media, the potential equations can be written in the form of the Telegraph equation, and the (2.7) Lorentz gauge condition modifies according to (2.14).

3. The wave equation and its solutions

We have already examined the particular solutions of Eq. (2.1) and (2.11), the retarded potentials. For the complete solution of the equations, the general solutions of the homogenous equations are also necessary. These equations jointly can be written as

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \,, \tag{3.1}$$

where $\boldsymbol{\psi}$ denotes one of followings: A_1, A_2, A_3, Φ and $c^2 = \frac{1}{\varepsilon\mu}$. Eq. (3.1) is called the wave equation. It is easy to see that in case of homogeneous isotropic insulators, a wave equation can be directly deduced for the field strengths ($\psi \rightarrow E_1, E_2, E_3, H_1, H_2, H_3$). The wave equations are present in other phenomena (acoustic, seismic) as well, then ψ denotes e.g., pressure, density or displacement. In homogenous media the c quantity in Eq. (3.1) is constant. For the clear solution of the equation both initial- and boundary conditions need to be set. Finding a solution that satisfies these condition is usually a quite challenging mathematical task. To simplify the solution let's assume that the source in the homogenous space is extremely far away, then we get the plane wave solution.

3.1 The plane wave solution of the wave equation

We get a particular solution of the wave equation (3.1) with the transformation of the independent variables

$$u = \omega t - (K_1x_1 + K_2x_2 + K_3x_3) \tag{3.2}$$

$$v = \omega t + K_1x_1 + K_2x_2 + K_3x_3 \,, \tag{3.3}$$

where ω, K_1, K_2, K_3 are real constants. Based on (3.2) and (3.3) it can be seen that

$$\frac{\partial}{\partial t} = \omega\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right), \frac{\partial}{\partial x_j} = K_j\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right),$$

and thus

$$\frac{\partial^2}{\partial t^2} = \omega^2\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2, \frac{\partial^2}{\partial x_j^2} = K_j^2\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial u}\right)^2, \quad \text{j=1,2,3.} \tag{3.4}$$

The $(\quad)^2$ sign indicates that the differentiation inside the parenthesis needs to be done twice.

Taking into consideration Eq. (3.4), the wave equation leads to

$$(K_1^2 + K_2^2 + K_3^2)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)^2 \psi - \frac{\omega^2}{c^2}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)^2 \psi = 0 \,.$$

If the equation

$$K_1^2 + K_2^2 + K_3^2 = \frac{\omega^2}{c^2} \tag{3.5}$$

is fulfilled, then

$$\left\{\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)^2 - \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)^2\right\}\psi = 0$$

or otherwise

$$4\frac{\partial^2\psi}{\partial u\partial v} = 0\ . \tag{3.6}$$

This equation can be trivially satisfied, if we take the function $\psi(u, v)$ in the following form

$$\psi(u, v) = f_1(u) + f_2(v)\ , \tag{3.7}$$

where f_1, f_2 are arbitrary functions that can be differentiated at least twice. The solution of the wave equation (3.7) is called the d'Alambert-solution.

With the notation $k^2 = \frac{\omega^2}{c^2}$, Eq. (3.5) can be written as

$$\frac{K_1^2}{k^2} + \frac{K_2^2}{k^2} + \frac{K_3^2}{k^2} = 1\ .$$

Based on this the $\vec{e}$ unit vector can be introduced with the following components

$$e_1 = \frac{K_1}{k}, e_2 = \frac{K_2}{k}, e_3 = \frac{K_3}{k}$$

By using this, Eq. (3.2) and (3.3) can be written in the form of

$$u = \omega t - k\vec{e}\vec{r}$$

$$v = \omega t + k\vec{e}\vec{r}$$

or otherwise

$$u = \omega t - \vec{k}\vec{r} \tag{3.8}$$

$$v = \omega t + \vec{k}\vec{r}\ , \tag{3.9}$$

where $\vec{k} = k\vec{e}$. So the d'Alambert-type solution of the wave equation is

$$\psi(x_1, x_2, x_3, t) = f_1\left(\omega t - \vec{k}\vec{r}\right) + f_2(\omega t + \vec{k}\vec{r}) \tag{3.10}$$

where f_1, f_2 are arbitrary functions, and between k the equation

$$k^2 = \frac{\omega^2}{c^2} \tag{3.11}$$

is fulfilled.

The (3.8) particular solution can be generalized easily. If

$$\psi_1 = f_1^{(1)}\left(\omega_1 t - \vec{k}\vec{r}\right) + f_2^{(1)}\left(\omega_1 t + \vec{k}\vec{r}\right)$$

$$\psi_2 = f_1^{(2)}\left(\omega_2 t - \vec{k}\vec{r}\right) + f_2^{(2)}\left(\omega_2 t + \vec{k}\vec{r}\right)$$

are the two independent solutions of the wave equation, then because of the linearity of the equation

$$\psi = \psi_1 + \psi_2$$

also satisfies the wave equation. Based on this the equation

$$\psi = \sum_{j=1}^{\infty}\left[f_1^{(j)}\left(\omega_j t - \vec{k_j}\vec{r}\right) + f_2^{(j)}\left(\omega_j t - \vec{k_j}\vec{r}\right)\right] \tag{3.12}$$

is also a solution. If the parameter ω is continuously distributed in the $[-\infty, \infty]$ interval, then as the superposition of the particular solutions the

$$\psi = \int_{-\infty}^{\infty} f_1\left(\omega t - \vec{k}\vec{r}\right)d\omega + \int_{-\infty}^{\infty} f_2\left(\omega t + \vec{k}\vec{r}\right)d\omega$$

functions produced with integration are also solution of Eq. (3.1). Since the wave equation contains time- and space coordinates derivatives, in the argument of the f_1, f_2 functions the ω parameter can appear separately

$$\psi = \int_{-\infty}^{\infty} f_1(u,\omega)d\omega + \int_{-\infty}^{\infty} f_2(v,\omega)d\omega \ , \qquad (3.13)$$

where we applied the notations (3.8) and (3.9).

The (3.12) and (3.13) generalization can also be done with the k_1, k_2, k_3 parameters:

$$\psi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f_1(u,\omega,k_1,k_2,k_3) + f_2(u,\omega,k_1,k_2,k_3)]\, d\omega dk_1, dk_2, dk_3 \ . \qquad (3.14)$$

The solution in (3.10) at fixed time gives the ψ function's (which characterizes the physical state) constant values on the

$$k\vec{e}\vec{r} = constant \ or \ \vec{e}(\vec{r} - r_0) = 0$$

surface, which is the equation of a plane ($\overrightarrow{r_0}$ is a position vector pointing to a fixed point of the space). Therefore, the function given in (3.10) in a general sense describes a plane wave. If in the description of a specific phenomenon from the space coordinates only x_1 plays a role, then (3.10) can be written as

$$\psi(x_1, t) = f_1(\omega t - kx_1) + f_2(\omega t + kx_1) \ . \qquad (3.15)$$

The f_1 function denotes the wave propagating in the positive direction on the $x_1$axis and the f_2 function denotes the wave propagating in the negative direction of x_1 axis. Since the coordinate system can be chosen arbitrary, f_1 *and* f_2 play the same role. Therefore in the followings it is sufficient just to deal with the f_1 part of the solution.

Until now the f_1 denoted an arbitrary chosen function. However, when dealing with waves, we usually come across functions that are periodic in time and space. For example:

$$\psi = \psi_0 \cos(\omega t - \vec{k}\vec{r} - \varphi) \quad , \qquad (3.16)$$

where ψ_0 is the constant amplitude, φ is the initial phase. The (3.16) function is the monochromatic plane wave solution of the wave equation. At a fixed t_0 time in the positions with the same $\alpha_0 = \omega t_0 - k\vec{e}\vec{r} - \varphi$ phases in the space they are located on the plane denoted by the equation:

$$\vec{e}(\vec{r} - \overrightarrow{r_0}) = 0 \ .$$

($\vec{e}\overrightarrow{r_0} = \omega t_0 - \varphi_0 - \alpha_0$), $\vec{e}$ is the normal unit vector of the plane wavefront.

If the period is T, then according to (3.16) the points of the wavefront related to t_0, at $t_0 + T$ time are in the same physical state, the phase however

$$\omega(t_0 + T) = \vec{k}\vec{r} - \varphi = \alpha_0 + 2\pi \ ,$$

from where

$$\omega = \frac{2\pi}{T} . \tag{3.17}$$

The ω constant in the (3.16) monochromatic plane wave solution is connected to time periodicity, it is called angular frequency.

In t_0 fixed according to (3.16) infinitely many planes can be found where the physical state is the same (the value of α is the same). The distance of two adjacent wavefronts is the wavelength which is characterized by the spatial periodicity. On the wavefront then the phase difference is 2π. If the adjacent plane's points are given by

$$\overrightarrow{r'} = \vec{r} - \lambda\vec{e} ,$$

then the phase

$$\omega t_0 - k\vec{e}\overrightarrow{r'} - \varphi = \alpha_0 + 2\pi ,$$

from where

$$k = \frac{2\pi}{\lambda} . \tag{3.18}$$

The constant k in (3.16) therefore is in connection with the spatial periodicity, and is called wavenumber and $\vec{k} = k\vec{e}$ is the wavenumber vector.

The wavefront is propagating in space. Directing the normal vector of the surface parallel with the propagation, with δt time the points of the

$$\alpha = \omega t - k\vec{e}\vec{r} - \rho$$

surface characterized by phase will move to the points of the wavefront

$$\alpha = \omega(t + \delta t) - k\vec{e}(\vec{r} + \delta\vec{r}) - \rho .$$

(During the propagation of the wavefront α remains unchanged). We interpret the $\delta\vec{r}$ displacement vector so that it is parallel with the $\vec{e}$ vector. Then $\vec{e}\delta\vec{r} = |\delta\vec{r}|$ and thus

$$\omega\delta t - k|\delta\vec{r}| = 0 ,$$

from where the velocity of propagation, the phase velocity is

$$v_f = \frac{|\delta\vec{r}|}{\delta t} = \frac{\omega}{k} .$$

Comparing this result with Eq. (3.11) we can see that $v_f = c$, which means that the c constant in the wave equation gives the phase velocity of plane waves propagating in infinite space. The function given in (3.16) is the real part of the complex function

$$\psi = \psi_0 e^{i(\omega t - \vec{k}\vec{r} - \varphi)} , \tag{3.19}$$

which is mathematically also a solution of the wave equation according to (3.10). To efficiently utilize the tools of the complex function theory, it is advisable to use the complex function (3.19) instead of (3.16). Obviously only the real part of complex expressions describing physical quantities have physical meanings.

Introducing

$\widehat{\psi_0} = \psi_0 e^{-i\varphi}$

complex amplitude, (3.19) can be written as

$$\psi = \widehat{\psi_0} e^{i(\omega t - \vec{k}\vec{r})} \quad . \tag{3.20}$$

This is the monochromatic plane wave solution of the wave equation in complex form. Based on (3.12) the generalized plane wave solution can be written in the form of

$$\psi = \sum_{j=1}^{\infty} \hat{\psi}_0^{(j)} e^{i(\omega_j t - \vec{k}\vec{r})} \quad .$$

Starting from the d'Alambert type solution (3.13) of the wave equation, its general form can be written through its particular solution (3.20) as:

$$\psi = \int_{-\infty}^{\infty} \widehat{\psi_0}(\omega) e^{i(\omega t - \vec{k}\vec{r})}\, d\omega \quad . \tag{3.21}$$

This equation shows the importance of the monochromatic plane wave solution, because it indicates that any wave phenomenon that varies arbitrarily over time (e.g. pulse) can be constructed as the superposition of monochromatic plane waves. Eq. (3.21) is also the Fourier-integral solution of the wave equation. This can be further generalized according to Eq. (3.14):

$$\psi = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \psi_0\left(\omega, \vec{k}\right) e^{i(\omega t - \vec{k}\vec{r})} d\omega d^3\vec{k} \,, \tag{3.22}$$

where $d^3\vec{k} = dk_1 dk_2 dk_3$.

Equation (3.11) shows the simple relationship between the angular frequency ω and the wavenumber k:

$$\omega = ck \,,$$

where c=constant. Generally the equations which give the $\omega = \omega(k)$ relation are called dispersion relations. The dispersion equations can also be complex and the ω and k quantities in them are not necessarily real either. Later we will see this for example in the description of attenuation of waves in space and time.

If the c^2 quantity in the wave equations is complex, then if Eq. (3.5) is fulfilled then as the solution of the wave equation we get function (3.7) again. According to Eq. (3.11) with the complex k wavenumber, (3.10), (3.16), (3.19) and (3.21) are still solution of Eq. (3.1). If $k = i\underline{a}$, then based on (3.19)

$$\psi = \psi_0 e^{i[\omega t - (b - i\underline{a})\vec{e}\vec{r} - \varphi]}$$

or otherwise

$$\psi = \psi_0 e^{-\underline{a}(\vec{e}\vec{r})} e^{i(\omega t - b\vec{e}\vec{r} - \varphi)} \quad .$$

If $\underline{a} \ll b$, this equation describes a wave with $v_f = \frac{\omega}{b}$ phase velocity that exponentially decreases its amplitude. The amplitude decreases from the initial $\psi_0$value to $\frac{\psi_0}{e}$, while the wave propagates $d = \frac{1}{\underline{a}}$ distance in the medium.

The d distance is the penetration depth. So the imaginary part of the complex wavenumber characterizes the attenuation of the wave, the $\underline{a}$ quantity is called the absorption coefficient.

3.2 The spherical wave solution of the wave equation

It often occurs that the examined wave phenomenon shows spherical symmetry (e.g. field of an isotopically transmitting point source). Then in spherical coordinate system $\psi = \psi(r,t)$ and

$$\Delta\psi(r) = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2}\ .$$

Introducing the $\psi_1 = r\psi$ notation, the wave equation (3.1) will have the form:

$$\frac{\partial^2\psi_1}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2\psi_1}{\partial t^2} = 0\ . \tag{3.23}$$

Taking into consideration that the

$$\frac{\partial^2\psi}{\partial x_1^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

one-dimensional wave equation's solution according to (3.15) can be written as:

$$\psi = (x_1, t) = f_1(\omega t - kx_1) + f_2(\omega t + kx_1)\ .$$

Therefore with the $x_1 \to r$ substitution (provided that (3.11) is fulfilled) the solution of (3.23) can be directly written as:

$$\psi_1(r,t) = f_1(\omega t - kr) + f_2(\omega t + kr)\ .$$

Thus the spherical wave solution of the wave equation is

$$\psi(r,t) = \frac{1}{r}f_1(\omega t - kr) + \frac{1}{r}f_2(\omega t + kr)\ . \tag{3.24}$$

As we saw in (3.15) the plane wave solution, the two particular solutions (the wave propagating along the x_1 axis to the right or left direction) physically had the same weight. However the difference between the two particular solutions in the spherical wave solution (3.24) is physically very important. The $\frac{1}{r}f_2$ particular solution describes a spherical wave propagating (divergent) out of the origin (source). This solution has a direct physical meaning. The $\frac{1}{r}f_2$ particular solution describes a spherical wave propagating into (convergent) the origin from the infinity. This should be transmitted from a spherical surface which has an infinite radius (as a source). Obviously this solution is physically not possible. However mathematically it is a solution of the wave equation and for the solution of some problems we use spherical wave series expansion.

The monochromatic spherical wave solution of the wave equation can be written as:

$$\psi = \frac{\psi_0}{r}\cos(\omega t - kr - \varphi) \tag{3.25}$$

or in a complex form:

$$\psi = \frac{\hat{\psi}_0}{r}e^{i(\omega t - kr)}\ , \tag{3.26}$$

where

$\hat{\psi}_0 = \psi_0 e^{-i\varphi}$

complex amplitude. In (3.25) and (3.26) the constants ω and k are the same as for the plane wave solution (3.17) and (3.18). The points of the spherical surface with the same phase

$\alpha = \omega t - kr$,

with δt time later will be on the wavefront

$\alpha = \omega(t + \delta t) - k(r + \delta r)$,

therefore

$\omega\delta t - k\delta r = 0$.

The propagation velocity of the wavefront – the phase velocity – thus

$v_f = \frac{\delta r}{\delta t} = \frac{\omega}{k}$.

So based on (3.11) $v_f = c$ again.

The superposition of the (3.26) monochromatic spherical wave solutions also satisfies the wave equation:

$$\psi(r,t) = \sum_{j=1}^{\infty} \frac{\psi_{0j}}{r} e^{i(\omega_j t - k_j r)} \ . \tag{3.27}$$

In this case we have a composite spherical wave solution. If the parameter ω is continuously distributed in the $[-\infty, \infty]$ interval, then (3.27) can be written in the more general form:

$$\psi(r,t) = \int_{-\infty}^{\infty} \frac{{\psi_0}^{(\omega)}}{r} e^{i(\omega t - k(\omega) r)} d\omega \ . \tag{3.28}$$

In this solution the ${\psi_0}^{(\omega)}$ function is arbitrary. However it is obvious that physically ${\psi_0}^{(\omega)}$ is in a relationship with the source of the wave.

3.3 Wave propagation in weakly inhomogeneous medium – the Eikonal equation

So far we have dealt with wave propagation in homogenous medium. We mentioned as a special feature of the plane wave solution, that the wavefront is an infinite plane surface, and the direction of wave propagation is constant (the wave propagates in a straight line). In practice the wavefront is always infinite and the path of the wave is usually bent because of the inhomogeneity of the medium and thus the wavefront is not plane either. In the followings we deal with the solution of the wave equation in inhomogeneous medium.

The time dependency of the field parameters separated in the form $e^{i\omega t}$

$\psi = \varphi(x_1, x_2, x_3)\, e^{i\omega t}$

from the wave equation we get the amplitude equation:

$\Delta\varphi + \frac{\omega^2}{v^2}\varphi = 0$,

where because of the inhomogeneity of the medium $v = v(x_1, x_2, x_3)$. Introducing the

$n(x_1, x_2, x_3) = \frac{c_0}{v}$

refractive index, the amplitude equation can be written in the form of:

$$\Delta\varphi + n^2 k_0^2 \varphi = 0 \, , \qquad (3.29)$$

where c_0 is a velocity dimension constant, and $k_0 = \frac{\omega}{c_0}$ is wavenumber like constant. In case of electromagnetic waves c_0 is usually the speed of light in vacuum, for other wave phenomena (seismic, acoustic) c_0 is the phase velocity at a given point of the medium. Thus in weakly inhomogeneous medium the n refractive index is of unit magnitude.

The solution of the Eq. (3.29) is not known for arbitrarily inhomogeneous medium. However, if the medium is weakly inhomogeneous, then we can derive an approximate solution. The medium is weakly inhomogeneous if the refractive index does not change considerably in the order of the wavelength, meaning:

$$|grad\ n|\lambda \ll n$$

Then we have a good reason to believe that the solution of Eq. (3.29) differs only slightly from the function we have for homogenous medium:

$$\varphi = \varphi_0 e^{ik_0 n\vec{e}\vec{r}} \, .$$

The difference may have two aspects, first the amplitude of the wave will depend on location, and secondly the wavefront is bent. Let's look for the solution in the following form:

$$\varphi(x_1, x_2, x_3) = u(x_1, x_2, x_3) e^{ik_0 W(x_1, x_2, x_3)} \, , \qquad (3.30)$$

where u is the amplitude function and W is the Eikonal function.

Due to the weak inhomogeneity of the medium, we expect the function u to be a slowly varying function of location and the W=constant surface not to differ much from the plane. These conditions can be expressed with the substitution of function (3.30) to (3.29). After the substitution we get the equation:

$$k_o^2 u[n^2 - (grad\ W)^2] + ik_o[2(grad\ W, grad\ u) + u\Delta W] + \Delta u = 0 \, , \qquad (3.31)$$

from where with the λ→0 borderline case, we get the Eikonal equation:

$$(grad\ W)^2 = n^2 \, . \qquad (3.32)$$

For optical phenomena the λ→0 borderline case leads to the field of geometric optics. It is known that in this approximation the propagation of light is described in the form of rays. The rays are the orthogonal trajectories of the W= constant wavefronts. The (3.32) Eikonal-equation creates a relationship between the W Eikonal function and the refractive index. Thus both the W function and the rays are defined by the distribution of the refractive index. Therefore (3.32) is the basic equation of geometric optics.

The condition for the fulfillment of the λ→0 borderline case can be given more accurately. In seismic, the waves are on the order of 10 meters, thus the λ→0 borderline case obviously means something else than in optics or in acoustics. Physically we have the λ→0 borderline case, if in Eq. (3.31) the member multiplied by k^2 dominates, meaning:

$$k_o n^2 u \gg 2\,(grad\ u, grad\ W) \, . \qquad (3.33)$$

$$k_o n^2 \gg \Delta W \quad . \qquad (3.34)$$

$k_o^2 n^2 u \gg \Delta u$. (3.35)

The (3.33) inequality expresses an orders of magnitude comparison, therefore the scalar multiplication on the right side can be substituted with the multiplication of the absolute values. Thus by using Eq. (3.32) we get the

$$k_o n u \gg 2|grad\, u|$$

relation. Or because of $k_o n = k = \frac{2\pi}{\lambda}$

$$|grad\, u|\lambda \ll u \, .$$

This inequality expresses the limitation that the amplitude function needs to be a slowly varying function of location, so it does not vary much in the order of the wavelength.

For the understanding of the (3.34) inequality let's suppose that $W = W(x_1)$. Then (3.32) will have the form of:

$$\frac{dW}{dx_1} = n \, ,$$

and the $W(x_1)$ plane curve's radius of curvature can be defined by:

$$R = \frac{(1+\frac{dW}{dx_1})^{3/2}}{\frac{d^2W}{dx_1^2}} \, ,$$

from where

$$\Delta W = \frac{d^2W}{dx_1^2} = \frac{(1+n)^{3/2}}{R} \, .$$

With this the (3.34) inequality can be written in the form of:

$$k_o n \gg \frac{(1+n)^{\frac{3}{2}}}{n} \frac{1}{R} \, .$$

If the refractive index n≈ 1 then

$$\frac{(1+n)^{\frac{3}{2}}}{n} \approx 2 \, ,$$

then (3.36) leads to the

$$R \gg \frac{\lambda}{\pi}$$

Relation. Therefore (3.34) is a limitation for the curvature of the wavefront. It provides that the radius of curvature (compared to the wavelength) of the wavefront is large. A similar condition can be written for the amplitude surface based in (3.35).

If the (3.33) and (3.35) conditions are fulfilled then we arrive to the approximation of λ→0 geometric optics, which means that both seismic and acoustic waves can be described by rays. The basic task is the following: we solve the (3.32) Eikonal equation and form the orthogonal trajectories of the W=constant surfaces. These note the rays.

The (3.32) Eikonal equation can also be written as:

$$gradW = n\vec{e} \,, \tag{3.37}$$

where $\vec{e}$ is the normal of the W standard surface – the unit vector pointing in the direction of the propagation of the wave. Based on (3.37) with the scalar multiplication with $\vec{e}$ we get the equation:

$$\frac{dW}{ds} = n \,,$$

where ds is the unit arc along the ray. After conversion we can give a new meaning to the W Eikonal function.

$$dW = nds = c\frac{ds}{v} \;.$$

Since $dt = \frac{ds}{v}$ is the time it takes for the ray to travel in the medium ds distance, the $dW = cdt$ is the unit distance, which it would travel during this time in vacuum (electromagnetic wave) or in the vicinity of the reference point (acoustic and seismic wave).

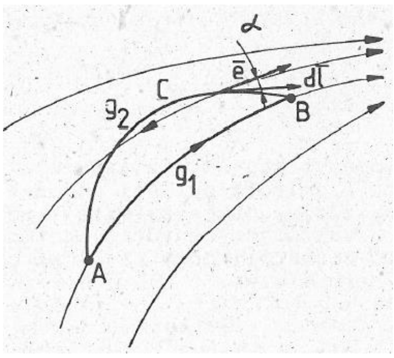

Figure 2.

The dW quantity is the unit optic distance, thus

$$W = \int_{P_o}^{P} nds$$

is the optical distance calculated for the ray's $\widehat{P_oP}$ path. Since rot grad W = 0 therefore based on (3.37) the

$$rot\ (n\vec{e}) = 0$$

equation can be written as well. Let's form the surface bordered by g_1 *and* g_2 curves according to Figure 2. We take the $g_1$curve along a ray, and g_2 is an arbitrary curve which differs slightly from g_1. Integrating the $rot(n\vec{e})$ function on the ABC surface and using the Stokes theorem

$$0 = \int_{ABC} rot(n\vec{e})d\vec{F} = \oint_g \; n\vec{e}dl$$

or otherwise

$$\int_{A/g_1}^{B} n\vec{e}\, d\vec{l} + \int_{B/g_2}^{A} n\vec{e}d\vec{l} = 0 \;\;.$$

The g_1 curve runs on the real ray path, thus $\vec{e}d\vec{l} = ds$, in turn on the g_2 curve $\vec{e}d\vec{l} = dl\ cos\alpha$. Swapping the limits in the second integral we get the equation:

$$\int_{A/g_1}^{B} nds + \int_{B/g_2}^{A} n\, cos\alpha\; dl = 0$$

from where because of cos$\alpha \leq 1$we get the

$$\int_{A/g_1}^{B} nds \leq \int_{\frac{A}{g_2}}^{B} ndl$$

result. This expresses that the optical path length calculated on the real ray path is always smaller than any other optical path calculated on adjacent curves. In other words, on the real ray path the optical path is the shortest. This is Fermat's principle.

4. Electromagnetic waves

Electromagnetic waves transmitted from artificial or natural sources are useful tools of the exploration of geological structures. For geophysical applications, primarily the wave propagation in the conducting medium needs to be studied. In this chapter we will discuss the electromagnetic waves propagating in infinite, homogenous medium and along infinite conducting half-space when the source is extremely far away. Then we will derive the fields of electric and magnetic dipoles transmitting in homogenous conductors and insulators. In order to show the similarities and differences between the properties of electromagnetic waves propagating in conductors and insulators, we will shortly summarize the most important facts about insulators.

4.1 Electromagnetic waves in homogenous, isotropic infinite insulator

Through Maxwell's discovery (He added displacement current to the electric current term in Ampère's Circuital Law) the structure of the system of basic equations describing the electromagnetic field transformed in such a way that wave equations can be derived from them. Maxwell realizing this predicted the theoretical existence of electromagnetic waves.

In non-conducting medium (insulator) with the assumption $\rho = 0$, taking the curl of Eq (1.1) we get the

$$rot\ rot\ \vec{H} = grad\ div\ \vec{H} - \Delta\vec{H} = \frac{\partial}{\partial t} rot\ \vec{E}$$

equation, which based on (1.2) and the material equations leads to the homogenous wave equation:

$$\Delta\vec{H} - \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} = 0\ . \qquad (4.1)$$

Similarly, for the electric field strength:

$$\Delta\vec{E} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0\ . \qquad (4.2)$$

The monochromatic plane wave solution of the equations based on (3.19) can be written directly as:

$$\vec{E} = \overrightarrow{E_0} e^{i(\omega t - \vec{k}\vec{r} - \varphi_1)}\ . \qquad (4.3)$$

$$\vec{H} = \overrightarrow{H_0} e^{i(\omega t - \vec{k}\vec{r} - \varphi_2)}\ . \qquad (4.4)$$

These functions do not satisfy the Maxwell equations directly. According to (1.3) in case of $\rho = 0$

$$div\vec{E} = 0\ . \qquad (4.5)$$

In the functions (4.3) and (4.4) the space coordinates have the form of $\vec{k}\vec{r} = k_1 x_1 + k_2 x_2 + k_3 x_3$ therefore

$$\frac{\partial E_1}{\partial x_1} = -ik_1 E_1, \frac{\partial E_2}{\partial x_2} = -ik_2 E_2, \frac{\partial E_3}{\partial x_3} = -ik_3 E_3$$

that is,

$div\vec{E} = -i\vec{k}\vec{E}$

Thus based on (4.5) we get the

$\vec{k}\overrightarrow{E_0} = 0$ or $\vec{e}\overrightarrow{E_0} = 0$

equation. The (4.3) function, which satisfies the wave equation, satisfies the (1.3) Maxwell equation only if $\vec{e}\perp\overrightarrow{E_0}$. Similarly, based on (1.4) and from (4.4) we get the condition $\vec{e}\overrightarrow{H_0} = 0$ or $\vec{e}\perp\overrightarrow{H_0}$.

Based on the (1.2) Maxwell equation and (4.3), (4.4)

$rot\vec{E} = -i\omega\mu\vec{H}$.

Since

$$rot\vec{E} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ E_1 & E_2 & E_3 \end{vmatrix} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -ik_1 & -ik_2 & -ik_3 \\ E_1 & E_2 & E_3 \end{vmatrix} = -i\vec{k}\text{x}\vec{E} \qquad (4.6)$$

therefore

$\vec{H} = \frac{k}{\omega\mu}\vec{e}\text{x}\vec{E}$

or because of $k = \omega\sqrt{\varepsilon\mu}$

$\vec{H} = \sqrt{\frac{\varepsilon}{\mu}}(\vec{e}\text{x}\vec{E})$

from where $\varphi_1 = \varphi_2$ and

$$\vec{H}_0 = \sqrt{\frac{\varepsilon}{\mu}}\left(\vec{e}\text{x}\overrightarrow{E_0}\right) . \qquad (4.7)$$

Between the magnetic and electric filed strength vectors there are no phase difference, their amplitude vectors are perpendicular not just to each other but to the unit vector $\vec{e}$ pointing to the direction of propagation as well. Based on (4.7)

$$H_0 = \sqrt{\frac{\varepsilon}{\mu}}E_0 \ . \qquad (4.8)$$

The energy density of the electric field

$W_E = \frac{1}{2}\varepsilon E^2$

and for the magnetic field the energy density is

$W_M = \frac{1}{2}\mu H^2$.

Based on (4.8) it can be seen that

$$W_M = W_E \ . \qquad (4.9)$$

Thus the energy density of the field is $W = W_M + W_E = \varepsilon E^2$.

The energy current density vector of the electromagnetic field is

$\vec{S} = \vec{E}\text{x}\vec{H}$

based on (4.7)

$$\vec{S} = cw\vec{e} \tag{4.10}$$

where $c = \frac{1}{\sqrt{\varepsilon\mu}}$

4.2 Electromagnetic waves in homogenous, isotropic infinite conductor

The most important properties of the propagation of electromagnetic waves in conductors can be derived in the plane wave approximation. The infinite plane wavefront is an ideal borderline case, which is fulfilled in practice satisfactorily, if the specific size of the finite wavefront and the radius of curvature of the surface are very large compared to the wavelength. The advantage of the plane wave approximation is that the field parameters can be defined with simple mathematical tools.

In homogenous conductor the differential Ohm's law has the form of $\vec{J} = \gamma\vec{E}$, where γ is the scalar conductivity which is independent of location.

With this the curl of the (1.1) Maxwell equation can be written as

$$rot\ rot\ \vec{H} = grad\ div\ \vec{H} - \Delta\vec{H} = \gamma\ rot\vec{E} - \frac{\partial}{\partial t} rot\ \vec{D}\ \ . \tag{4.11}$$

Using the $\vec{D} = \varepsilon\vec{E}, \vec{B} = \mu\vec{H}$ material equations (ε, μ are constants) and with the $\rho = 0$ assumption the

$div\vec{D} = 0,\ div\ \vec{B} = 0$

Maxwell equations, based in (1.2) and (4.11) can be brought to the

$$\Delta\vec{H} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} - \gamma\mu\frac{\partial\vec{H}}{\partial t} = 0 \tag{4.12}$$

form. Taking the curl of Eq. (1.2) with a similar process, for the electric field strength we get the equation

$$\Delta\vec{E} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} - \gamma\mu\frac{\partial\vec{E}}{\partial t} = 0\ \ . \tag{4.13}$$

So in homogenous, isotropic conductors the field strength vectors satisfy the (4.12), (4.13) telegraph equations. Looking for the plane wave solution of these equations, the time dependency of the $\vec{E}, \vec{H}$ field parameters can be assumed in the form of $e^{i\omega t}$.

Then the

$\frac{\partial\vec{E}}{\partial t} = \frac{1}{i\omega}\frac{\partial^2\vec{E}}{\partial t^2},\ \ \frac{\partial\vec{H}}{\partial t} = \frac{1}{i\omega}\frac{\partial^2\vec{H}}{\partial t^2}$

relationships are met, which (4.12), (4.13) formally can be transformed to wave equations:

$$\Delta\vec{H} - \frac{1}{v^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \qquad (4.14)$$

$$\Delta\vec{E} - \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0\ , \qquad (4.15)$$

where $\frac{1}{v^2} = \varepsilon\mu(1 - i\tau),\ \ \tau = \frac{\gamma}{\varepsilon\omega}$. Here we would like to mention that the medium, describing phenomena showing $e^{i\omega t}$ time dependency, can be characterized by the complex dielectric constant

$$\varepsilon' = \varepsilon(1 - i\tau)$$

introduced in equations (4.14), (4.15). Thus we also introduce the $v = \frac{1}{\sqrt{\varepsilon'\mu}}$ complex phase velocity. The solutions of Eq. (4.14), (4.15) can be found with the method presented in chapter 3, however the k wavenumber introduced with the

$$\frac{\omega^2}{v^2} = k^2 \qquad (4.16)$$

equation, now is complex

$$k = b - ia\ . \qquad (4.17)$$

The monochromatic plane wave solutions written based on (3.19) will look as:

$$\vec{E} = \vec{E}_0 e^{i(\omega t - k\vec{e}\vec{r} + \varphi_1)} \qquad (4.18)$$

$$\vec{H} = \vec{H}_0 e^{i(\omega t - k\vec{e}\vec{r} + \varphi_2)} \qquad (4.19)$$

(Since with the used complex method

$$\Delta\vec{E} = -k^2\vec{E},\ \ \Delta\vec{H} = -k^2\vec{H} \qquad (4.20)$$

the (4.18), (4.19) functions substituted into the equations (4.14), (4.15), we indeed get an equation which correspond with Eq. (4.16)

$$k^2 = \omega^2\varepsilon\mu(1 - i\tau)\ . \qquad (4.21)$$

From the equations (4.17) and (4.21) for the imaginary and real parts of the complex wave number we get the

$$b^2 - a^2 = \omega^2\varepsilon\mu$$

$$2ab = \omega^2\varepsilon\mu\tau$$

equations, from where

$$b^4 - \omega^2\varepsilon\mu b^2 - \frac{(\omega^2\varepsilon\mu\tau)^2}{4} = 0\ .$$

From the equation’s roots

$$b_1^2 = \frac{\omega^2\varepsilon\mu}{2}\left(1 + \sqrt{1 + \tau^2}\right)$$

$$b_1^2 = \frac{\omega^2\varepsilon\mu}{2}\left(1 + \sqrt{1 + \tau^2}\right)$$

only $b_1$is real, therefore the solutions of (4.21) are

$$b = \pm\sqrt{\frac{\omega^2\varepsilon\mu}{2}(1+\sqrt{1+\tau^2})} \tag{4.22}$$

$$a = \pm\sqrt{\frac{\omega^2\varepsilon\mu}{2}(-1+\sqrt{1+\tau^2})}\ . \tag{4.23}$$

In these equations the – and + signs are chosen depending on the coordinate system. Giving the electric field strength in the following form:

$$\vec{E} = \overrightarrow{E_0}e^{i[\omega t-(b-ia)(\vec{e}\vec{r})+\varphi_1]}$$

we can see that if a > 0

the function

$$\vec{E} = \vec{E}_0 e^{-a(e\ r)}e^{i[\omega t-b(\vec{e}\vec{r})+\varphi_1]} \tag{4.24}$$

in case of $\vec{e}\vec{r} > 0$ an attenuating and in case of $\vec{e}\vec{r} < 0$ describes a wave that exponentially increases in amplitude. Thus we get the physically acceptable solution in (4.22), (4.23) in case of $(\vec{e}\vec{r})$>0 by choosing the (+) and in case of $(\vec{e}\vec{r}) < 0$ choosing the (-) sign.

In one dimensional case $\vec{e} = (1,0,0)$ (4.24) can be written as

$$\vec{E} = \vec{E}_0 e^{-ax_1}e^{i(\omega t-bx_1+\varphi_1)}\ .$$

In case of a << b, this solution describes a wave propagating in the direction of the x_1 axis with $v = \frac{\omega}{b}$ phase velocity, which amplitude attenuates by the e^{-ax_1} function. If d denotes the distance, along which the amplitude measured at the $x_1 = 0$ position attenuates by $\frac{1}{e}$ then ad=1, so

$$d = \frac{1}{\omega\sqrt{\varepsilon\mu}\sqrt{\frac{1}{2}(-1+\sqrt{1+(\frac{\gamma}{\varepsilon\omega})^2})}}\ . \tag{4.25}$$

The d distance is characteristic of the attenuation of the wave. Approximately it gives the wave's depth of penetration in conducting medium. It is also called skin depth. As it can be seen in (4.25) it mainly depend on frequency and conductivity.

The $v = \frac{\omega}{b}$ phase velocity beside the material properties also depends on the frequency as it can be seen in the expression:

$$v = \frac{1}{\sqrt{\varepsilon\mu}}\frac{1}{\sqrt{\frac{1}{2}(1+\sqrt{1+(\frac{\gamma}{\varepsilon\omega})^2})}}\ . \tag{4.26}$$

So wave propagation in conducting medium is dispersive.

The (4.18), (4.19) functions satisfy the (4.12), (4.13) telegraph equations derived from the Maxwell equations, if the complex k=b-ia wavenumber is the solution of Eq. (4.21). However, the field strengths also have to satisfy the Maxwell equations. It is obvious that it means further restrictive conditions.

Eq. (4.18) satisfies the $div\,\vec{E} = 0$ equation if the equation

$$k\left(\vec{e}\vec{E}_0\right) = 0$$

is fulfilled, which leads to the condition $\vec{e} \perp \vec{E}_0$ even though k is complex. Similarly, the $div\,\vec{H} = 0$ equations leads to $\vec{e} \perp \vec{H}_0$. So the field strength vectors even for electromagnetic waves propagating in conducting medium are perpendicular to the direction of wave propagation.

From the (1.2) Maxwell equation, using (4.6) we get the condition

$$\vec{H} = \frac{k}{\omega\mu}\vec{e}\mathrm{x}\vec{E}\ . \tag{4.27}$$

Using (4.22) and (4.23), the complex wave number can be written with Euler's formula as:

where

$$\sqrt{a^2 + b^2} = \omega\sqrt{\varepsilon\mu}\sqrt[4]{1+\tau^2}$$

$$\delta = arctg\,\left(\frac{a}{b}\right) = arctg\sqrt{\frac{-1+\sqrt{1+\tau^2}}{1+\sqrt{1+\tau^2}}}\ . \tag{4.28}$$

With this based on (4.27) we get to the result:

$$\overrightarrow{H_0} = \sqrt{\frac{\varepsilon}{\mu}\sqrt{1+\tau^2}}(\vec{e}\mathrm{x}\overrightarrow{E_0})e^{i(\varphi_1-\varphi_2-\delta)}\ ,$$

from where

$$\overrightarrow{H_0} = \sqrt{\frac{\varepsilon}{\mu}}\sqrt[4]{1+\tau^2}(\vec{e}\mathrm{x}\overrightarrow{E_0}) \tag{4.29}$$

and

$$\varphi_1 - \varphi_2 = \delta\ \ . \tag{4.30}$$

The electric and magnetic field strength vectors of the electromagnetic waves propagating in conducing medium are perpendicular to each other and to the direction of propagation as well, meaning that the waves are transverse. Between the amplitudes the relationship is the following:

$$H_0 = \sqrt{\frac{\varepsilon}{\mu}\sqrt{1+\tau^2}}E_0\ \ . \tag{4.31}$$

According to (4.30) between the field strengths there is δ phase difference, which based on (4.28) can take the values $0 \le \delta \le \frac{\pi}{4}$, while the conductivity varies (and thus the $\tau = \frac{\gamma}{\varepsilon\mu}$ as well) on the $(0,\infty)$ interval. So that magnetic field strength in homogenous conductors always delays compared to the electric field strength. The phase difference is 45° at most.

In case of $\varphi_1 = 0$

$$\vec{E} = \vec{E}_0 e^{i(\omega t - k\vec{e}\vec{r})} \tag{4.32}$$

$$\vec{H} = \vec{H}_0 e^{i(\omega t - k\vec{e}\vec{r} - \delta)}\ \ . \tag{4.33}$$

Now let’s calculate the energy density of the electromagnetic wave propagating in a conductor

$W_{EM} = \frac{1}{2}\varepsilon(Re\,\vec{E})^2 + \frac{1}{2}\mu(Re\,\vec{H})^2$

where

$Re\,\vec{E} = \frac{1}{2}(\vec{E} + \vec{E}^*),\; Re\,\vec{H} = \frac{1}{2}(\vec{H} + \vec{H}^*)$

denote the real parts of the (4.18), (4.19) complex electric field strengths, $*$ is the notation of complex conjugate. The (4.32) energy density is a fast varying function of time and space, with measurements we usually determine its time average. But since in the expression

$(Re\,\vec{E})^2 = \frac{1}{4}(\vec{E}\vec{E} + \vec{E}^*\vec{E}^* + 2\vec{E}\vec{E}^*)$

$\vec{E}\vec{E}$ depends on time $e^{2i\omega t}$, $\vec{E}^*\vec{E}^*$ depends on time as $e^{-2i\omega t}$, during time averaging only the time independent $\vec{E}\vec{E}^*$ member remains:

$(Re\,\vec{E})^2 = \frac{1}{2}\vec{E}_0^*$.

So the average of the energy density by using

$W_{EM} = \frac{1}{2}(Re\,\vec{E})^2 + \frac{1}{2}(Re\,\vec{H})^2 = \frac{1}{4}(\varepsilon E_0^2 + \mu H_0^2)$

or by using (4.31) is:

$W_{EM} = \left(1 + \sqrt{1+\tau^2}\right)W_E$,

where

$W_E = \frac{1}{2}\varepsilon(Re\,\vec{E})^2$ is the average of the electric energy density. Introducing the $W_M = \frac{1}{2}\mu(Re\,\vec{H})^2$ average magnetic energy density we can see that

$$\frac{W_M}{W_E} = \sqrt{1+\tau^2} \;, \tag{4. 34}$$

which means that in electromagnetic waves propagating in conducting medium, the magnetic energy density is always greater than the electric energy density.

The results derived for electromagnetic waves propagating in conducting medium can be further studied in two boundary cases. The first $\gamma \to 0$ $(\tau \to 0)$ boundary case leads to the known equations of insulators. In this case in (4.14), (4.15) $v = \sqrt{\varepsilon\mu}$, so there is no dispersion. According to (4.22) $k = b = \omega\sqrt{\varepsilon\mu}$, and according to (4.23) a=0 which means that the wave attenuates. From Eq. (4.28) we get $\delta = 0$, which means that there is no phase difference between the filed strengths vectors, and Eq. (4.29) in case of $\tau = 0$ returns Eq. (4.7).

The other boundary case is the high conductivity boundary case $\tau = \frac{\gamma}{\varepsilon\omega} \gg 1$. As we have seen in the subchapter 1.4, this is the Quasi-Stationary or low frequency approximation: $\omega \ll \frac{\gamma}{\varepsilon}$, $\frac{\gamma}{\varepsilon}$ is the relaxation time. We can also mention that in this case the displacement current density is much smaller than the current density. The $\tau \gg 1$ condition, for limestone which has the lowest electric conductivity is approximately fulfilled up to $10^6 Hz$ (for other rocks even higher). Since the depth of penetration at $10^6 Hz$ in limestone is $\approx 15m$ (for other rocks even smaller), the frequency used in measurements needs to be much smaller than 1MHz. Thus the $\tau \gg 1$ condition is always fulfilled in geophysical application. Then the electric and magnetic field strengths based on (4.32) and (4.33) can be written as:

$$\vec{E} = \vec{E}_0 e^{i(\omega t - k\vec{e}\vec{r})}$$

$$\vec{H} = \vec{H}_0 e^{i(\omega t - k\vec{e}\vec{r} - \delta)} \ ,$$

where

$$k = \sqrt{\frac{1}{2}\gamma\mu\omega}(1-i) \ . \qquad (4.35)$$

(So the attenuation is not weak: b=a), and based on (4.28) $\delta = \frac{\pi}{4}$. In this bordierline case the penetration depth of the electromagnetic wave is

$$d = \sqrt{\frac{2}{\gamma\mu\omega}} \ .$$

The relationship between the amplitudes of the field strengths is given based on (4.31) by the equation:

$$H_0 = \sqrt{\frac{\gamma}{\mu\omega}} E_0 \ .$$

Between the time averages of the magnetic and electric energy densities the following relation is fulfilled:

$$\frac{w_M}{w_E} \approx \tau \ .$$

The functions (4.32), (4.33) which we have got as the solutions of the telegraph equation, give the field strengths of the electromagnetic plane waves propagating in conductive medium. The main property of the waves, that they attenuate and are dispersive. We can experience the attenuation of electromagnetic waves in everyday life for example when the radio quiets down in tunnels and in reinforced concreate buildings. The results we have got can also be confirmed by some optical examples. Glass is a good insulator and in it electromagnetic waves (light as well) do not attenuate much, therefore the glass is transparent. Metals are good conductors, therefore they are opaque to electromagnetic waves, especially for light. There are some counterexamples which are at first glance puzzling, ebonite, Bakelite and caprolactam are good insulators, so we would expect them to be transparent, but they are not. On the other hand, even though salt crystal is a relatively good conductor, it is not opaque.

This contradiction is solved by the fact that in our derivations the “material constants” ε, μ, γ were assumed to be constants, but they are a function of frequency. This frequency dependence should not be overlooked, for example if we would like to utilize our results which we got from 50Hz current, in optical frequencies of $10^{15} Hz$. Thus for example the ebonite, Bakelite etc. known as insulators above $10^9 Hz$ behave as good conductors and salt is an insulator in optical frequency.

4.3 Electromagnetic waves along infinite conductive half-space

Up until now we have dealt with wave propagation in infinite media. We assumed the wave source to be infinitely far away, thus we did not have to deal with the initial and boundary conditions. However, in geophysical application usually we have a layered medium. Then the solutions of the wave equations at the interfaces have to satisfy the boundary conditions shown in subchapter 1.1. The easiest assumption which makes it possible to study the most important properties of waves propagating in layered medium is to take an infinite conductor contacting a nonconducting half space. Let $x_3 > 0$ half-space be an insulator $\varepsilon, \mu, (\gamma = 0)$, the $x_3 < 0$ half-space be a conductor, with the $\varepsilon, \mu, \gamma \neq 0$ material properties and let's assume that the wave is arriving along $x_1$from a source infinitely far away. Then the parameters of the fields depend on time and on the x_1 coordinate as $e^{i(\omega t - k x_1)}$, and the x_2 coordinate does not affect the description of the phenomenon ($\frac{\partial}{\partial x^2} = 0$), so

$$\vec{E}(x_1, x_3, t) = \vec{E}^{*}(x_3) e^{i(\omega t - k x_1)} \; . \tag{4.36}$$

$$\vec{H}(x_1, x_3, t) = \vec{H}^{*}(x_3) e^{i(\omega t - k x_1)} \; . \tag{4.37}$$

Using the material equations as well with this the (1.1) and (1.2) Maxwell equations can be written in the form of:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -ik & 0 & \frac{\partial}{\partial x_3} \\ H_1 & H_2 & H_3 \end{vmatrix} = (\gamma + i\omega\varepsilon) \begin{vmatrix} E_1 \\ E_2 \\ E_3 \end{vmatrix} \; .$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -ik & 0 & \frac{\partial}{\partial x_3} \\ E_1 & E_2 & E_3 \end{vmatrix} = -i\omega\mu \begin{vmatrix} H_1 \\ H_2 \\ H_3 \end{vmatrix} \; .$$

By expounding the equations, we get two independent system of equations:

$$\frac{\partial H_2}{\partial X_3} = -(\gamma + i\omega\varepsilon) E_1 \tag{4.38}$$

$$ikH_2 = -(\gamma + i\omega\varepsilon) E_3 \tag{4.39}$$

$$ikE_3 + \frac{\partial E_1}{\partial X_3} = -i\omega\mu H_2 \; . \tag{4.40}$$

and

$$\frac{\partial E_2}{\partial X_3} = i\omega\mu H_1 \tag{4.41}$$

$$ikE_2 = i\omega\mu H_3 \tag{4.42}$$

$$ikH_3 + \frac{\partial H_1}{\partial X_3} = (\gamma + i\omega\varepsilon) E_2 \; . \tag{4.43}$$

From the two equation groups it is enough just to solve one of them, e.g. the (4.38)-(4.40) system of equations, because from this the field parameters (H_2, E_1, E_3) →(E_2, H_1, H_3) and with the substitution of the material properties $-(\gamma + i\omega\varepsilon) \rightarrow i\omega\mu$, we get the (4.41)-(4.43) equations.

Expressing the $E_1^{*}(x_3), E_3^{*}(x_3)$ functions from the equations (4.38) and (4.39)

$$E_1^{*} = -\frac{1}{\gamma + i\omega\varepsilon} \frac{dH_2^{*}}{dx_3} \tag{4.44}$$

$$E_3^{\divideontimes} = -\frac{1}{\gamma + i\omega\varepsilon} H_2^{\divideontimes} \tag{4.45}$$

and substituting into (4.40) we get the equation

$$\frac{d^2 H_2^{\divideontimes}}{dx_3^2} + (k_\infty^2 - k^2) H_2^{\divideontimes} = 0\ , \tag{4.46}$$

where we introduced the

$$k_\infty^2 = -i\omega\mu(\gamma + i\omega\varepsilon) \tag{4.47}$$

notation. The general solution of the (4.46) differential equation can be simply written as:

$$H_2^{\divideontimes}(x_3) = Ae^{i\sqrt{k_\infty^2 - k^2}x3} + Be^{-i\sqrt{k_\infty^2 - k^2}x3}\ , \tag{4.48}$$

where A and B are constant of integration.

The expression $\sqrt{k_\infty^2 - k^2}$ returns two complex numbers, which are each other's complex conjugate. Both roots are suitable to describe the field strength. From now on we will use the one which has the positive imaginary part. Then in the $x_3 > 0$ half-space in (4.48) we have to choose B=0, because the expression $e^{i\sqrt{k_\infty^2 - k^2}x3}$ in case of $x_3 \to \infty$ approaches to infinity, but the field strength can only have a finite value. Similarly, in the $x_3 < 0$ half-space only in case of A=0 we get a regular solution. Thus we get to the

$$H_2^{\divideontimes}(x_3) = \begin{cases} Ae^{i\sqrt{k_o^2 - k^2}x_3} & x_3 > 0 \\ Be^{-i\sqrt{k_\infty^2 - k^2}x_3} & x_3 < 0 \end{cases}$$

result, where $k_0^2 = \omega^2\varepsilon\mu$. However, based on the (1.5) boundary condition

$$H_2(x_3 \to +0) = H_2(x_3 \to -0)\ ,$$

from where A=B, thus the (4.46) equation is regular in $\pm\infty$ and its solution on the $x_3 = 0$ plane fulfilling the boundary conditions as well:

$$H_2^{\divideontimes} = \begin{cases} Ce^{i\sqrt{k_o^2 - k^2}x_3} & x_3 > 0 \\ Ce^{-i\sqrt{k_\infty^2 - k^2}x_3} & x_3 < 0 \end{cases}\ . \tag{4.49}$$

Using this and according to (4.44) and (4.46)

$$E_1^{\divideontimes} = \frac{i\omega\mu}{k_\infty^2}\frac{dH_2^{\divideontimes}}{dx_3} = \begin{cases} -\frac{\omega\mu}{k_o^2}\sqrt{k_o^2 - k^2}\,Ce^{i\sqrt{k_o^2 - k^2}x_3} & x_3 > 0 \\ \frac{\omega\mu}{k_\infty^2}\sqrt{k_\infty^2 - k^2}\,Ce^{-i\sqrt{k_\infty^2 - k^2}x_3} & x_3 < 0 \end{cases} \tag{4.50}$$

and based on (4.45) we get the

$$E_3^{*} = \begin{cases} -\frac{\omega\mu k}{k_o^2} C e^{i\sqrt{k_o^2-k^2}x_3} & x_3>0 \\ -\frac{\omega\mu k}{k_\infty^2} C e^{-i\sqrt{k_\infty^2-k^2}x_3} & x_3<0 \end{cases} \quad (4.51)$$

result. The k wavenumber in the equations (4.49) -(4.51) is unknown. The (1.6) boundary condition

$$E_1(x_3 \to +0) = E_1(x_3 \to -0)$$

based on (4.50) on the $x_3 = 0$ plane, provides the fulfillment of the equation:

$$-\frac{1}{k_o^2}\sqrt{k_o^2 - k^2} = \frac{1}{k_\infty^2}\sqrt{k_\infty^2 - k^2} \ ,$$

from where

$$\frac{1}{k^2} = \frac{1}{k_o^2} + \frac{1}{k_\infty^2} \quad . \quad (4.52)$$

This equation is the dispersion relation of the electromagnetic waves propagating along infinite conductive half-space, which with the help of (4.47) can be written as:

$$\frac{1}{k^2} = \frac{1}{k_o^2}\left[1 + \frac{1}{1-i\tau}\right] ,$$

where $\tau = \frac{\gamma}{\varepsilon\omega}$.

In the high conductivity borderline case $\tau \gg 1$ and thus

$$k^2 \approx k_0^2 \ .$$

Then in the $x_3 > 0$ half-space, based on (4.49), (4.50), (4.51) we get to

$$H_2 = Ce^{i(\omega t - k_o x_1)}$$

$$E_1 = 0$$

$$E_3 = -\mu c \, Ce^{i(\omega t - k_o x_1)} \ ,$$

where, $c = \frac{\omega}{k_o}$. So in insulator we get a transversal electromagnetic wave propagating without attenuation nor dispersion with c phase velocity, which amplitude is independent from the x_3 coordinate. This solution equals the solution we have got for infinite insulator. Paradoxically, if the conducting half-space is an extremely good conductor ($\tau \gg 1$) then it has no effect to the insulator half-space.

In the $x_3 < 0$ half-space

$$|k_\infty^2| = k_o^2\tau = k^2\tau^2 \ ,$$

therefore $k_\infty^2 - k^2 \approx k_\infty^2$ and thus for the field strengths we get the equations:

$$E_1 = \frac{k_o}{k}\mu c C e^{-ik_\infty x_3} e^{i(\omega t - k_o x_1)} \ .$$

$$E_3 = \frac{k_o^2}{k^2}\mu c C e^{-ik_\infty x_3} e^{i(\omega t - k_o x_1)} \ .$$

$H = Ce^{-ik_\infty x_3}e^{i(\omega t - k_o x_1)}$.

However, based on (4.47) it is easy to see that in the high conductivity borderline case

$$k_\infty = \sqrt{\frac{\gamma\mu\omega}{2}}(-1+i)$$

and thus the above expression can be written in the form of:

$$E_1 = \frac{\mu c}{\sqrt{\tau}} C e^{\sqrt{\frac{\gamma\mu\omega}{2}}x_3} e^{i\left(\omega t - k_o x_1 + \sqrt{\frac{\gamma\mu\omega}{2}}x_3 + \frac{\pi}{4}\right)} \quad . \tag{4.52}$$

$$E_3 = i\frac{\mu c}{\sqrt{\tau}} C e^{\sqrt{\frac{\gamma\mu\omega}{2}}x_3} e^{i\left(\omega t - k_o x_1 + \sqrt{\frac{\gamma\mu\omega}{2}}x_3\right)} \quad . \tag{4.53}$$

$$H = Ce^{e^{\sqrt{\frac{\gamma\mu\omega}{2}}x_3}} e^{i\left(\omega t - k_o x_1 + \sqrt{\frac{\gamma\mu\omega}{2}}x_3\right)} \quad . \tag{4.54}$$

In the conducting half-space, the field strengths are exponentially decreasing as we get farther from the interface. The depth where it decreases to its $\frac{1}{e}$ part is the skin depth:

$$d = \sqrt{\frac{2}{\gamma\mu\omega}} \tag{4.55}$$

This distance gives the approximate penetration depth of electromagnetic waves into conducting medium. In the high conductivity borderline case the $\vec{H}\ and\ \vec{E}_1$ field strengths are perpendicular to each other and the phase difference between them is $\frac{\pi}{4}$.

The (4.52)-(4.54) expressions contain the space coordinates in the form of $-k_1x_1 + k_3x_3$, where $k_1 = k_o, k_3 = \sqrt{\frac{\gamma\mu\omega}{2}}$.

So the wave in the conducting half-space propagates in the (x_1x_3) plane. Denoting the angle between the direction of propagation and the x_3 axis with ϑ

$$tg\vartheta = \frac{k_1}{k_3} = \sqrt{\frac{2}{\tau}} \, .$$

Since $\tau \gg 1$, therefore $\vartheta \approx \sqrt{\frac{1}{\tau}}$. So the wave propagates almost parallel with the x_3 axis.

4.4 The field of the radiating electric diploe in infinite insulator

Up until now we neglected the source with the assumption that the electromagnetic waves arrive from the infinity. Therefore, we could use the plane wave approximation in the studied part of the space. Now let’s examine the field of the electric point dipole! Let’s assume that dipole radiates in homogenous, isotopic infinite insulator medium, so we do not have to deal with boundary conditions. To start with a more general case, we will start by examining the field of a dipole which is pointlike. For the sake of simplicity, and because in applications the volume charge density does not play an important role, let $\rho = 0$.

Based on the equations (1.8) the electric displacement vector can be written as:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} \ , \tag{4.56}$$

where $\vec{P}$ electric polarization vector. With this the (1.1)-(1.4) Maxwell equations can be written as:

$$rot\ \vec{H} = \overrightarrow{J_P} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} \tag{4.57}$$

$$rot\ \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.58}$$

$$div\left(\varepsilon_o \vec{E}\right) = \rho_p \tag{4.59}$$

$$div\ \vec{B} = 0\ , \tag{4.60}$$

where we introduced the

$$\overrightarrow{J_P} = \frac{\partial \vec{P}}{\partial t} \tag{4.61}$$

polarization density and the

$$\rho_p = -div\ \vec{P} \tag{4.62}$$

polarization charge density. (The medium is an insulator, so the current density is zero.)

In the potential equations derived in the 2.2 the sources were the conduction current density and the volume charge density. In their place in the equations (1.1)-(1.4), we have the $\overrightarrow{J_P}$ polarization density and ρ_p polarization charge density in (4.57)-(4.60) and instead of the vector $\vec{D}$ we have $\varepsilon_o \vec{E}$.

With the same method as in chapter 2. with the equations

$$\vec{B} = rot\vec{A},\ \ \vec{E} = -grad\Phi - \frac{\partial \vec{A}}{\partial t}\ , \tag{4.63}$$

we can introduce the electromagnetic potentials. From the equations (4.57) and (4.59) we get the

$$\Delta\vec{A} - \varepsilon_o \mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \overrightarrow{J_P} \tag{4.64}$$

$$\Delta\Phi - \varepsilon_o \mu \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho_p}{\varepsilon_o} \tag{4.65}$$

potential equations, provided that the

$$div\ \vec{A} + \varepsilon_o \mu \frac{\partial \Phi}{\partial t} = 0$$

Lorentz gauge condition is fulfilled. This last equation can be trivially fulfilled if we introduce the $\vec{Z}$ Hertz vector with the

$$\Phi = -div\ \vec{Z} \tag{4.66}$$

$$\vec{A} = \varepsilon_o \mu \frac{\partial \vec{Z}}{\partial t} \tag{4.67}$$

equations. With this the equations (4.64), (4.65) taking into account (4.61), (4.62), will have the form of:

$$-div\left(\Delta\vec{Z} - \varepsilon_o\mu\frac{\partial^2\vec{Z}}{\partial t^2} + \frac{\vec{P}}{\varepsilon_o}\right) = 0$$

$$\varepsilon_o\mu\frac{\partial}{\partial t}\left(\Delta\vec{Z} - \varepsilon_o\mu\frac{\partial^2\vec{Z}}{\partial t^2} + \frac{\vec{P}}{\varepsilon_o}\right) = 0 \ ,$$

where for the Hertz vector we get the d'Alambert differential equation

$$\Delta\vec{Z} - \varepsilon_o\mu\frac{\partial^2\vec{Z}}{\partial t^2} = -\frac{\vec{P}}{\varepsilon_o} \ . \tag{4.68}$$

The solution of the equations can be written directly as seen in 2.2

$$\vec{Z}(x_1, x_2, x_3, t) = \frac{1}{4\pi\varepsilon_o}\int_{V'}\frac{P(x_1, x_2, x_3, t-\frac{R}{c})}{R}dV' \ , \tag{4.69}$$

where $= \sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2 + (x_3' - x_3)^2}$, and c is the velocity of the electromagnetic wave.

With the (4.69) Hertz vector, any $\vec{P}$ dipole moment distribution's field can be calculated. For the fixed pointlike dipole in the r_o point of the field a density function can be assigned by the Dirac function δ

$$\vec{P} = (\vec{r}; t') = \vec{P}(t')\delta\left(\overrightarrow{r'} - \overrightarrow{r_o}\right) ,$$

where $\vec{P}(t')$ is the dipole moment of the pointlike dipole. According to (4.69) then the Hertz vector is given by the integral

$$\vec{Z}(r, t) = \frac{1}{4\pi\varepsilon_o}\int_{V'}\frac{\vec{P}(t-\frac{R}{c})\delta\left(\overrightarrow{r'} - \overrightarrow{r_o}\right)}{R}dV' \ ,$$

where $R = \left|\vec{r} - \overrightarrow{r'}\right|$. The integral can be calculated based on the properties if the Dirac δ

$$\vec{Z}(\vec{r}, t) = \frac{\vec{P}(t-\frac{R}{c})}{4\pi\varepsilon_o R} , \tag{4.70}$$

where $R = |\vec{r} - \overrightarrow{r_o}|$.

So the field of the pointlike dipole with the (4.70) Hertz vector and based on the equations (4.66), (4.67) and (4.63) can be written as:

$$\vec{B} = rot(\varepsilon\mu\frac{\partial\vec{Z}}{\partial t}) \tag{4.71}$$

$$\vec{E} = -grad\left(-div\vec{Z}\right) - \frac{\partial}{\partial t}(\varepsilon_o\mu\frac{\partial\vec{Z}}{\partial t}) \ . \tag{4.72}$$

Using the equation

$$rot\ rot\ \vec{Z} = grad\ div\ \vec{Z} - \Delta\vec{Z} \ , \tag{4.73}$$

(4.72) can be written as:

$$\vec{E} = rot\,rot\,\vec{Z} + \Delta\vec{Z} - \varepsilon_o\mu\frac{\partial^2\vec{Z}}{\partial t^2}$$

or by (4.68)

$$\vec{E} = rot\,rot\,\vec{Z} - \frac{\vec{P}}{\varepsilon_o}\delta(\vec{r} - \vec{r_o})\ .$$

Using these equation, the electric field strength can be calculated at any point of the space. At the location of the pointlike dipole this is usually not necessary. Outside the dipole however $\delta(\vec{r} - \vec{r_o}) = 0$ thus

$$\vec{E} = rot\,rot\,\vec{Z}(\vec{r} \neq \vec{r}_o)\ . \qquad (4.74)$$

The equations (4.71) and (4.74) give the pointlike dipole's field placed at the $\vec{r}_o$ point. Let's place the dipole in the spherical coordinate system's origin and direct it to the $\vartheta = 0$ direction of the polar axis. Then the components of the dipole moment vector $P_r = P\,cos\vartheta,\ P_\vartheta = -P\,sin\vartheta,$ $P_\vartheta = 0$ where $P = |\vec{P}|$. The Hertz vector in spherical coordinate system can be written as:

$$\vec{Z} = (r,\vartheta,t) = \left\{\frac{P\left(t-\frac{r}{c}\right)cos\vartheta}{4\pi\varepsilon_o r}, -\frac{P\left(t-\frac{r}{c}\right)sin\vartheta}{4\pi\varepsilon_o r}, 0\right\}\ . \qquad (4.75)$$

(Choosing the dipole's direction according to Figure 3., the dipole's field is obviously independent from the φ coordinate.)

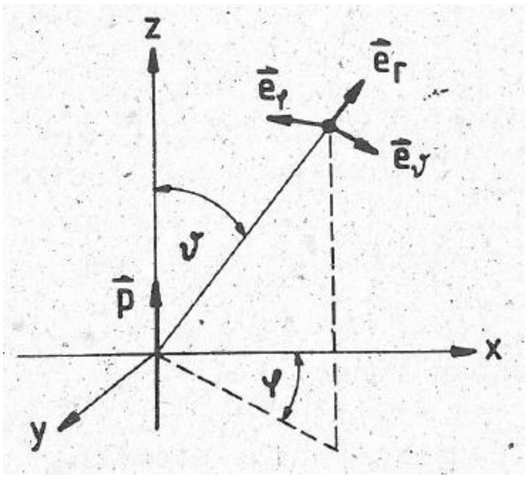

Figure 3.

The $\vec{Q}(r,\vartheta)$ vector function's curl on spherical coordinate system can be calculated with equations:

$$(rot\ Q)_r = \frac{1}{r\,sin\vartheta}\frac{\partial}{\partial\vartheta}(Q\varphi sin\vartheta)$$

$$(rot\ Q)_\vartheta = -\frac{1}{r}\frac{\partial}{\partial r}(rQ\varphi) \qquad (4.76)$$

$$(rot\ Q)_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rQ\vartheta) - \frac{\partial Q_r}{\partial\vartheta}\right]\ .$$

Using the (4.75) expression of the Hertz vector, with the $\vec{Q} = \vec{Z}$ substitution, according to (4.71) and (4.76):

$$B_r = 0, B_\vartheta = 0, B_\varphi = \frac{\mu sin}{4\pi}(\frac{P'}{r^2} + \frac{P"}{cr}) \ , \qquad (4.77)$$

where $P' = \frac{\partial P}{\partial t}, P" = \frac{\partial P\prime}{\partial t}$. With the $\vec{Q} = rot\,\vec{Z}$ substitution, based on (4.72) and (4.76) the electric field strength can be written as:

$$E_r = \frac{cos\vartheta}{2\pi\varepsilon_o}\left(\frac{P}{r^3} + \frac{P'}{cr^2}\right), E_\vartheta = \frac{sin\vartheta}{4\pi\varepsilon_o}\left(\frac{P}{r^3} + \frac{P'}{cr^2} + \frac{P"}{c^2 r}\right), E_\varphi = 0 \ . \qquad (4.78)$$

The dipole moment's time dependency of the radiating dipole in the practically important cases can be taken in the form of $P_o e^{i\omega t}$. According to (4.75) in the studied points of the space, because of the retardation we get a $e^{i\omega(t-\frac{r}{c})}$ space and time dependency for the field quantities. (Because of the isotropy only the radial r coordinate is present, there are no $\vartheta\ and\ \varphi$ dependency). Then the (4.77), (4.78) solutions will have the following forms:

$$B_r = \frac{P_o cos\vartheta}{2\pi\varepsilon_o}\left(\frac{1}{r^3} + \frac{1\omega}{cr^2}\right) e^{i\omega\left(t-\frac{1}{c}\right)} \ . \qquad (4.79)$$

$$E_\vartheta = \frac{P_o sin\vartheta}{4\pi\varepsilon_o}\left(\frac{1}{r^3} + \frac{1\omega}{cr^2} - \frac{\omega^2}{c^2 r}\right) e^{i\omega\left(t-\frac{r}{c}\right)} \ . \qquad (4.80)$$

$$E_\varphi = 0,\ B_r = 0,\ B_\vartheta = 0 \quad . \qquad (4.81)$$

$$B_\varphi = \frac{P_0 \mu sin\vartheta}{4\pi}\left(\frac{i\omega}{r^2} - \frac{\omega^2}{cr}\right) e^{i\omega\left(t-\frac{r}{c}\right)} \ . \qquad (4.82)$$

This electromagnetic field can be divided into 3 parts based on their distance dependence:

$$\vec{E} = \vec{E}^{(o)} + \vec{E}^{(1)} + \vec{E}^{(2)} \ .$$

$$\vec{B} = \vec{B}^{(o)} + \vec{B}^{(1)} + \vec{B}^{(2)} \ .$$

where

$$\vec{E}^{(o)} = \frac{P_o}{2\pi\varepsilon_o r^3} e^{i\omega(t-\frac{r}{c})} \left\{cos\vartheta, \frac{1}{2} sin\vartheta, 0\right\} . \qquad (4.83)$$

$$\vec{B}^{(o)} = \{0,0,0\} \ . \qquad (4.84)$$

$$\vec{E}^{(1)} = \frac{i\omega P_o}{2\pi\varepsilon_o c r^3} e^{i\omega(t-\frac{r}{c})} \left\{cos\vartheta, \frac{1}{2} sin\vartheta, 0\right\} . \qquad (4.85)$$

$$\vec{B}^{(1)} = \frac{i\omega P_o}{2\pi r^2} e^{i\omega(t-\frac{r}{c})} \left\{0,0, \frac{1}{2} sin\vartheta\right\} . \qquad (4.86)$$

$$\vec{E}^{(2)} = -\frac{P_o \omega^2}{2\pi\varepsilon_o c^2 r} e^{i\omega(t-\frac{r}{c})} \left\{0, \frac{1}{2} sin\vartheta, 0\right\} . \qquad (4.87)$$

$$\vec{B}^{(2)} = -\frac{\omega^2 \mu P_o}{2\pi c r} e^{i\omega(t-\frac{r}{c})} \left\{0,0, \frac{1}{2} sin\vartheta\right\} . \qquad (4.88)$$

The (4.83), (4.84) equations mean the remaining part of the solution in the ($\omega \to 0$) borderline case, therefore we call this field static, or near zone. The latter expression is justified by the comparison of the order of $\vec{E}^{(o)}, \vec{E}^{(1)}, \vec{E}^{(2)}$ field strengths. The

$$\frac{E^{(o)}}{E^{(1)}} \approx \frac{c}{r\omega} \approx \frac{\lambda}{r}$$

relation shows that in long distance compared to the wavelength ($\lambda \ll r$) the $E^{(o)} \ll E^{(1)}$ "static" field can be neglected, it only plays an important role in the field near the dipole. Similarly

$$\frac{E^{(1)}}{E^{(2)}} \approx \frac{\lambda}{r}, \frac{B^{(1)}}{B^{(2)}} \approx \frac{\lambda}{r} \ ,$$

which means that each member is the previous member times $\frac{\lambda}{r}$. Therefore in a long distance from the dipole only the $E^{(2)}, B^{(2)}$ field play roles, therefore the (4.87), (4.88) formulas give the field parameters in the far zone or in the wave zone. The naming "wave zone" is justified by the fact that $E^{(2)} \perp B^{(2)}$ and (4.87), (4.88) are monochromatic spherical waves. The (4.85), (4.86) expressions of $E^{(1)}, B^{(1)}$ dominate in the transition zone.

In practice we create electric dipoles by flowing current between to electrodes. If

$I = I_o e^{i\omega t}$, then because of $I = \frac{dQ}{dt}$

$$Q = \frac{I_o}{i\omega} e^{i\omega t}$$

So

$$\vec{P} = Q\delta\vec{l} = \frac{I_o \delta\vec{l}}{i\omega} e^{i\omega t} \ ,$$

where $|\delta\vec{l}|$ is the spacing of the electrodes. The field of the dipole thus can be calculated with:

$$\vec{E} = \left\{\frac{I_o \delta l \cos\vartheta}{2\pi\varepsilon_o i\omega}\left(\frac{1}{r^3} + \frac{i}{cr^2}\right), \frac{I_o \delta l \sin\vartheta}{4\pi\varepsilon_o i}\left(\frac{1}{r^3} + \frac{i\omega}{cr^2} - \frac{\omega^2}{c^2 r}\right), 0\right\} e^{i\omega(t-\frac{r}{c})} \ . \tag{4.89}$$

$$\vec{B} = \left\{0, 0, \frac{I_o \delta l \sin\vartheta}{4\pi}\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right)\right\} e^{i\omega(t-\frac{r}{c})} \ . \tag{4.90}$$

4.5 Field of the electric dipole radiating in infinite conductor

For the calculation of the field of the electric dipole radiating in infinite conductor we can again use the (4.56) expression. Then from the (4.57)-(4.60) Maxwell equations only (4.57) will modify

$$rot\vec{H} = \vec{J}_P + \gamma\vec{E} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} \ . \tag{4.91}$$

Introducing the electromagnetic potentials according to (4.63), from (4.91) we get the

$$\Delta\vec{A} - \varepsilon_o \mu \frac{\partial^2 \vec{A}}{\partial t^2} - \gamma\mu \frac{\partial \vec{A}}{\partial t} = -\mu\vec{J}_P + grad\left(div\,\vec{A} + \varepsilon_o \mu \frac{\partial \Phi}{\partial t} + \gamma\mu\Phi\right)$$

Equation. As explained in 2.3 let's set a modified Lorentz gauge condition corresponding to (2.14)

$$div\,\vec{A} + \varepsilon_o \mu \frac{\partial \Phi}{\partial t} + \gamma\mu\Phi = 0 \ . \tag{4.92}$$

Then this equation leads to the

$$\Delta\vec{A} - \varepsilon_o \mu \frac{\partial^2 \vec{A}}{\partial t^2} - \gamma\mu \frac{\partial \vec{A}}{\partial t} = -\mu\vec{J}_P \tag{4.93}$$

telegraph equation.

Let's introduce the $\vec{Z}$ Hertz vector so that

$\Phi = -div\,\vec{Z},\ \vec{A} = \varepsilon_o \mu \frac{\partial \vec{Z}}{\partial t} + \gamma \mu \vec{Z}$.

Then the (4.92) condition is trivially fulfilled.

Assuming that the time dependency has the form of: $e^{i\omega t}$ we can use the substitution

$\vec{Z} = \frac{1}{i\omega}\frac{\partial \vec{Z}}{\partial t}$.

Thus for the vector potential we get the equation

$$\vec{A} = \varepsilon' \mu \frac{\partial \vec{Z}}{\partial t}, \qquad (4.94)$$

where $\varepsilon' = \varepsilon_o (1 - i\frac{\gamma}{\varepsilon_o \omega})$. Similarly we can bring (4.93) to the form of

$$\Delta\vec{A} - \varepsilon' \mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}_P$$

Or based on (4.94) and (4.61)

$$\varepsilon' \mu \frac{\partial}{\partial t}\left\{\Delta\vec{Z} - \varepsilon' \mu \frac{\partial^2 \vec{Z}}{\partial t^2} + \frac{\vec{P}}{\varepsilon'}\right\} = 0 ,$$

from where for the Hertz vector we get the equation

$$\Delta\vec{Z} - \varepsilon' \mu \frac{\partial^2 \vec{Z}}{\partial t^2} = \frac{\vec{P}}{\varepsilon'} . \qquad (4.95)$$

Comparing the equations (4.95), (4.68), and (4.94), (4.67) we find that the equations describing the field of the electric dipole radiating in infinite conductor only differs from the equations derived for insulators that in the place of ε_o constant, we have the complex ε' dielectric constant. So it is not necessary to repeat the deductions we did for the insulator. With the $\varepsilon_o \to \varepsilon'$ substitution the field parameters based on (4.89), (4.90)

$$\vec{E} = \left\{\frac{I_o \delta l\, cos\vartheta}{2\pi\varepsilon' i\omega}\left(\frac{1}{r^3} + \frac{i}{c' r^2}\right), \frac{I_o \delta l\, sin\vartheta}{4\pi\varepsilon' i\omega}\left(\frac{1}{r^3} + \frac{i\omega}{c' r^2} - \frac{\omega^2}{c'^2 r}\right), 0\right\} e^{i\omega(t - \frac{r}{c'})} \qquad (4.96)$$

$$\vec{B} = \left\{0, 0, \frac{I_o \delta l\, sin\vartheta}{4\pi}\left(\frac{1}{r^2} - \frac{i\omega}{c' r}\right)\right\} e^{i\omega(t - \frac{r}{c'})} , \qquad (4.97)$$

where

$$c' = \frac{1}{\sqrt{\mu_o \varepsilon'}}$$

is the complex phase velocity. Introducing the $k = \frac{\omega}{c'}$ complex wavenumber, based on k=b-ia we can see that in the conducting medium the

$$e^{i\omega(t - \frac{r}{c'})} = e^{-ar} e^{i(\omega t - br)}$$

function gives the exponential attenuation of the field quantities.

4.6 Field of magnetic dipole radiating in infinite insulator

For the deduction of the field of the pointlike magnetic dipole we again start with field of the magnetic moment per unit volume. With the breaking down as (1.8)

$\vec{B} = \mu_o \vec{H} + \vec{M}$,

the Maxwell equations (assuming $\rho = 0$)

$$\mathrm{rot}\vec{E} = -\vec{J}_M - \mu_o \frac{\partial \vec{H}}{\partial t} \ . \tag{4.98}$$

$$\mathrm{rot}\vec{H} = \frac{\partial \vec{D}}{\partial t} . \tag{4.99}$$

$$\mathrm{div}\ (\mu_o \vec{H}) = \rho_M \ . \tag{4.100}$$

$$\mathrm{div}\,\vec{D} = 0 \ , \tag{4.101}$$

where

$$\vec{J}_M = \frac{\partial \vec{M}}{\partial t} .$$

$$\rho_M = -div\ \vec{M} \ .$$

With the different order of the equations it is even more obvious that the role of the electric and magnetic quantities is "interchanged". $\vec{J}_M$ (Magnetic polarization) current density from (4.98) replacing 1.1 and ρ_M from (4.100) plays a similar role as ρ volume charge density in 1.3. This analogy justifies, that following the method of the introduction of the electromagnetic potentials shown in 2., we can write the trivial solution of (4.101) as:

$$\vec{D} = -rot\vec{a} \ .$$

We call this quantity vector potential as well, however its role and meaning is different from the earlier introduced $\vec{A}$ quantity. Form the equation (4.99) using (4.102)

$$rot\ \left(\vec{H} + \frac{\partial \vec{a}}{\partial t}\right) = 0 \ , \tag{4.102}$$

from where

$$\vec{H} = -grad\ \varphi - \frac{\partial \vec{a}}{\partial t} \ . \tag{4.103}$$

Here φ is the newly introduced scalar potential. The equation (4.98) based on the (4.102), (4.103) definitions of $\vec{a}$ vector and φ scalar potential

$$\Delta\vec{a} - \varepsilon\mu_o \frac{\partial^2 \vec{a}}{\partial t^2} = -\varepsilon\vec{J}_M + grad(div\vec{a} + \varepsilon\mu_o \frac{\partial \varphi}{\partial t}) \ ,$$

and (4.100) leads to the equation

$$\Delta\varphi - \varepsilon\mu_o \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho_M}{\mu} - \frac{\partial}{\partial t}(div\vec{a} + \varepsilon\mu_o \frac{\partial \varphi}{\partial t}) \ .$$

Again following the

$$div\vec{a} + \varepsilon\mu_o \frac{\partial \varphi}{\partial t} = 0 \tag{4.104}$$

Lorentz gauge condition we get the potentials equations:

$$\Delta\vec{a} - \varepsilon\mu_o \frac{\partial^2 \vec{a}}{\partial t^2} = -\varepsilon\vec{J}_M \tag{4.105}$$

$$\Delta\varphi - \varepsilon\mu_o \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho_M}{\mu} \tag{4.106}$$

The (4.104) condition can be trivially satisfied if we introduce the $\vec{Z}$ Hertz vector with the equations

$$\varphi = -div\vec{Z} \tag{4.107}$$

$$\vec{a} = \varepsilon\mu_o \frac{\partial \vec{Z}}{\partial t} \tag{4.108}$$

according to (4.66) and (4.67).

Based on (4.105) and (4.106) for the $\vec{Z}$ vector we again get the

$$\Delta\vec{Z} - \varepsilon\mu_o \frac{\partial^2 \vec{Z}}{\partial t^2} = -\frac{\vec{M}}{\mu_o}$$

equation. The solution of this based on 2.2

$$\Delta\vec{Z}(\vec{r}, t) = \frac{1}{4\pi\mu_o} \int_V \frac{\vec{M}(\vec{r}; t - \frac{R}{c})}{R} dV' \ , \tag{4.109}$$

where $R = \left|\vec{r'} - \vec{r}\right|$ and c is the velocity of the electromagnetic effect's propagation.

In case of pointlike magnetic dipole the $\vec{M}$ magnetization can be written with the help of the Dirac function δ as:

$$\vec{M}(r', t') = \vec{m}(t')\delta\left(\vec{r'} - \vec{r}_o\right) ,$$

where $\vec{r}_o$ is the position of the dipole. With this from (4.109) for the Hertz vector we get the

$$\vec{Z}(r, t) = \frac{\vec{m}(t - \frac{R}{c})}{4\pi\mu_o R} \tag{4.110}$$

equation. Based on (4.102,) (4.103) and (4.107), (4.108) for the field quantities we get the equations

$$\vec{D} = -\varepsilon\mu_o \, rot \frac{\partial \vec{Z}}{\partial t} \tag{4.111}$$

$$\vec{H} = rot\, rot\, \vec{Z} - \frac{\vec{m}}{\mu_o}\delta(r - r_o) ,$$

where we used the (4.73) expression. If we study the electromagnetic field outside the dipole ($\vec{r} \neq \vec{r_o}$), then the previous equation simplifies to

$$\vec{H} = rot\, rot\, \vec{Z} \ . \tag{4.112}$$

Compering the equations (4.110)-(4.112) with the equations we have got for the electric dipoles (4.70), (4.71) and (4.74) we can see that in the latter in the place of $\vec{B}$ if we write $-\vec{D}$, instead of $\vec{E}$ we write $\vec{H}$ vector, instead of ε_o we write μ_o, and instead of μ we write ε, then we get the equations deduced for the magnetic dipoles. Therefore we do not have to solve the equations (4.110)-(4.112), their solution can be written directly based on (4.79)-(4.82):

$$H_r = \frac{m_o cos\vartheta}{2\pi\mu_o}\left(\frac{1}{r^3} + \frac{1\omega}{cr^2}\right) e^{i\omega\left(t-\frac{r}{c}\right)} \ . \tag{4.113}$$

$$H_\vartheta = \frac{m_o sin\vartheta}{4\pi\mu_o}\left(\frac{1}{r^3} + \frac{i\omega}{cr^2} - \frac{\omega^2}{c^2 r}\right) e^{i\omega\left(t-\frac{r}{c}\right)} \ . \tag{4.114}$$

$$H_\varphi = 0,\ D_r = 0,\ D_\vartheta = 0 \ . \tag{4.115}$$

$$D_\varphi = -\frac{m_0 sin\vartheta}{4\pi}\left(\frac{i\omega}{r^2} - \frac{\omega^2}{c^2 r}\right) e^{i\omega\left(t-\frac{r}{c}\right)} \ . \tag{4.116}$$

According to the analogy it is obvious, that the field (4.113)-(4.116) is also divided into near- (static magnetic dipole's field), transition and far or wave zone.

In practice the magnetic dipole is a coil with flowing current, which dipole moment in case of a coil with just one loop is

$$m = \mu_o I\delta A \ \ ,$$

where δA the coils cross-section. If $I = I_o e^{i\omega t}$, then the electromagnetic field of the coil is given by:

$$\vec{H} = \frac{I_o\delta A}{2\pi}\left\{\left(\frac{1}{r^3} + \frac{i\omega}{cr^2}\right) cos\vartheta, \frac{1}{2}\left(\frac{1}{r^3} + \frac{i\omega}{cr^2} - \frac{\omega^2}{c^2 r}\right) sin\vartheta, 0\right\} e^{i\omega(t-\frac{r}{c})}$$

$$\vec{D} = \frac{\mu_o I_o\delta A}{4\pi}\left\{0,0,\left(\frac{i\omega}{r^2} - \frac{\omega^2}{c^2 r}\right) sin\vartheta\right\} e^{i\omega(t-\frac{r}{c})} \quad .$$